Dear AATM Colleagues,

Welcome to the Fall issue of OnCore. We thank the wonderful talented authors who took time to contribute their great ideas for stimulating student interest in mathematics and providing additional resources for teachers. As well as in the Spring issue, the last article is a collection of problems that we created for middle/high school students. All problems focus on Very Strange Numbers, some of which may be familiar, and others, most “strange”!!!

Subtracting “Take-Away” from Elementary Mathematics: What’s the Difference? addresses difficulties in understanding caused by teaching subtraction as the process of take-away rather than the process of identifying the difference. Problems from kindergarten through grade 6 mathematics are provided along with student solutions. Of interest is the fact that the problems lead, naturally, to the study of integers, particularly negative numbers.

Great Problems Often Lead to Bigger Ideas! describes problems/investigations for students in grade bands (K-2, 3-5, and 6-8) that employ line graphs, geoboards, Base 10 blocks, and hundreds chart, and have multiple solutions or solution methods. As the author points out, the problems “are not an end-in-and-of-themselves, but lead to bigger ideas.”

Ever wonder how authors create 100s of different Sudoku puzzles? How Different Sudokus Can be The Same shows ways that a 9-by-9 puzzle can be rearranged to create a different puzzle by applying various geometric transformations.

Merging Math Content with Applications: Increase Interest and Achievement presents seven applications of mathematics for middle and high school students. All require investigation, observation, brainstorming, conjecturing, data collection and analyses, and the use of various technologies. The contexts were selected based on student interest and recommendation.

Is it our responsibility to introduce high school students to contemporary mathematics? The gap between school mathematics and contemporary mathematics is described along with examples of both. Persuasive arguments in favor of including contemporary mathematics are presented.

From Euclid to Math Competitions to Math Classes highlights the importance of teaching problem solving (ala Euclid), and that math principles and facts are useful in that endeavor but not the foci. Instructional strategies are described, as well as interesting illustrative problems with solutions.

Very Strange Numbers provides descriptions of 13 different types of numbers (e.g., EMIRP, Happy, Unusual, Harmonic, Proth) that were found/invented by number theorists in current and past centuries. Each number type includes related problems, with solutions.

Enjoy these articles. Consider submitting articles about your favorite problems/projects.

With very best wishes,

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Subtracting “Take-Away” from Elementary Mathematics: What’s the Difference?

James A. Middleton

Abstract
The use of the term, “take-away” is shown to be problematic as a metaphor for subtraction in the elementary grades. Instead, emphasis should be placed on the concept of “difference”. Activities that exemplify this concept for teaching multi-digit arithmetic are provided, along with student work samples for a variety of age/ability levels.

When I was a graduate student at the University of Wisconsin, there were a number of research projects in the US trying to create tools and curricula that would help bring the mathematics of change—mathematics that lead up to the calculus—into elementary and middle-school classrooms. The idea was that increasing the number of students who experienced mathematics of change early in their educational experience would result in increased enrollment and performance in Calculus courses.

Calculus, the course, is certainly important for those planning to be STEM majors or enter careers in economics and other social sciences. But Calculus has long been criticized as a filter, keeping most able students from those careers, as opposed to a pump, maintaining or even increasing mathematical opportunities for students (Steen, 1988). Since that time, the role of data has eclipsed Calculus, the course, in its everyday and academic importance. Every student in the US system is now expected to gain sophisticated experience with statistics, probability, and “data science” - a loose association of data modeling fields that include computer science, machine learning, and computational mathematics (National Governors Association, 2010).

This shift in curricular priorities places an equally important emphasis on the mathematics of change, but our approach needs to be different. Instead of requiring facility with limits, derivatives and integrals, there are deeper, more fundamental concepts that need bolstering in the K12 curriculum, most of which are easier conceptually to understand, than traditional calculus. Two of these may seem almost trivial: Difference and Quotient.

I have studied children’s learning of these two concepts throughout my career and found that misconceptions associated with them prevent many students from gaining comfortable insight into the big ideas of algebra, statistics, and calculus. With these concepts in their grasp, students can reason through many problems that traditionally require calculus, but instead, they can use computational tools to approximate solutions to very close tolerances. This essentially makes elementary and middle school mathematics THE foundation for data-intensive careers in students’ futures. So to emphasize one of the key foundational ideas in school mathematics, presented in a way that is both simple for young children to grasp, but sophisticated in the ways in which it promotes the kind of thinking necessary for higher-level mathematics, we will take a deep dive into the concept of difference.
**Difference**

One of the first concepts students explore in preschool and kindergarten is that of *difference*. We typically introduce *difference* in counting situations like the following.

Problem 1: Devon has 7 apples. He gives 5 to Loretta. How many apples does Devon have left?

Problem 2: Devon has 7 apples. He gives some apples to Loretta. He now has 5 apples left. How many apples did he give to Loretta?

Problem 3: Loretta has 5 apples. Devon has 7 apples. How many more apples does Devon have than Loretta?

These three problems appear to be the same. Just subtract 5 from 7. But 40 years of research show that young children don’t think of difference as the operation “subtraction”. Instead, they tend to read story problems as real stories, acting-them-out in their imaginations, or use manipulatives or other tools (Carpenter & Moser, 1984).

As an example, for Problem 3, young children typically count Loretta’s 5 apples using fingers, tally marks, blocks or other counters, using one counter for each apple, and counting from 1 to keep track of the cardinality. Then, students count Devon’s apples in the same manner, stopping at 7. Assuming there are no counting errors, the student will then pair “apples” one-to-one, and then count the leftovers: “One, two. Devon has two more apples than Loretta.”

This strategy is typical. Just compare what is the same in the two sets, and then what is not the same. That is the difference. The other two problems can be reasoned through if one takes the literal mind of a child: Count seven apples, keep track of them with counters/objects, count 5 and then separate them from the 7 to show the change. Then count the leftovers. Or, count the 7, give some to Loretta, and count what remains. Guess and check until Devon has 5 remaining and then indicate that 2 is the difference.

The transition from this *direct modeling* to subtraction takes some time. To make things move faster, teachers often will introduce the concept of *take-away*: “Seven take away 2 equals 5.” While this is convenient, and for many situations that students encounter, it suffices. However, it is a poor metaphor for subtraction that distracts students from the underlying concept of difference. There are several reasons for this. With *take-away*, one cannot subtract a quantity greater than the minuend (the number that is being subtracted from). Five minus seven is not allowed with this metaphor. Moreover, practices like counting on—typically used in addition problems—are not allowable, because subtraction is equated in students’ minds with counting back. Counting back is much more difficult. To illustrate this, try “counting back” fluently from J to E in the alphabet.
I have a lovely video recording of a second grader, Dena, solving the following problem:

\[
\begin{array}{c}
53 \\
-27 \\
\end{array}
\]

She has no difficulty producing the answer:

Dena: “3 take away 7. That equals 4. And 5 take away 2 equals 3. That’s 34.”

Dena was then asked to solve the problem using Base 10 Blocks. She correctly counted out 5 tens and 3 ones. She then removed 2 tens from the 5 tens, and then stopped for a bit looking at the paper.

Interviewer: “What are you thinking?”

Dena: “I don’t have enough to take away 7.”

Interviewer: “What can you do to get enough?” With this prompt, Dena remembered that she could separate one 10 into 10 ones and use those.

Dena: “Thirteen take away 7 is 6. I don’t get it.”

Interviewer: “What don’t you get?”

Dena: “Well, 3 take away 7 is 4. 5 take away 2 is 3. I don’t get it.”

After a flummoxed moment, the interviewer asks Dena to explain her reasoning, and she repeats her procedure, subtracting in column fashion as before, and then comparing her procedure with the correct reasoning using the Base10 Blocks.

Figure 1. Dena’s correct use of Base 10 Blocks to illustrate 53-27 with regrouping.
Throughout the interview, Dena insisted that her procedure was correct, and implied that something had to be wrong with how she was using the Base10 Blocks. The take-away metaphor only allows lesser numbers to be subtracted from greater numbers. So, when she began learning multi-digit arithmetic, that misconception was carried with her. As well, her experiences were so focused on building procedural fluency (a good thing, in general) with the two-column format, her misconception with this format remained robust to her lovely proof of the problem using the concrete manipulatives. Contrast Dena’s case with that of Kwame solving the same problem:

![Figure 2. Kwame’s strategy for subtracting 27 from 53.](image)

“So, 3 minus 7 is 4*, and 5 minus 2 is 3 (zero added later). That is 30 minus 4, or 26.”

When asked to explain the star by the 4 in his solution, Kwame said, “It is different. Because 3 minus 7, you keep that (pointing at 4*) and subtract from 30.”

What Kwame demonstrates is the naturalness of negative numbers when students are freed of the burden of take-away. The difference between 3 and 7 is always 4 units. The issue for the learner is how to keep track of the direction. Kwame had committed the difference between 7 and 3 to memory, but students often count from 3 up to 7, or sometimes, count down from 7 to 3. It is 4 units regardless of their approach.

Kwame cleverly chose to represent what we know as negative 4 with a symbol he devised (*). Later, this can be transitioned to the standard negative sign. When I was a boy, we were taught to show a negative number with a raised minus like this, `-4`, so that it distinguished the signed numbers from the symbol for the operation of subtraction. So, 3 `-4` would clearly show 3 minus negative 4. One of my teachers even forbid us to use the word, “minus”. He insisted that we use “subtract” exclusively (e.g., “three subtract negative four”) so that the sign and the operation were clear in our heads. I am not sure what best practices really *should be* for denoting negative numbers when they arise in the second or third grade. The main issue is to not shy away from them. Build them into a classroom conversation about what the word “difference” means, mathematically.
The Common Core State Standards for School Mathematics (NGA&CCSSO, 2010) do not include integers as important content for students before Grade 6. Yet, in every 2nd grade lesson I have ever seen, where “multi-digit subtraction with regrouping” was the focus, at least two students invented some notation for negative numbers and used it with understanding. Moreover, when I became a worker in retail, I was taught to count change from the price of the item to the amount given to me. It was a natural way to think, unfettered with rules of regrouping: All you had to do was be able to count by 1s (cents), nickels, dimes, quarters, and dollars. Students early on begin to count by 1s, 2s, 5s and 10s—skip counting with conceptual understanding and good fluency. Why not build on this propensity?

This begs the question, if students can deal with early concepts of signed numbers, at least the integers, should we take it up or should we delay it until the curriculum says it is necessary?

You all know this is a rhetorical question. My perspective is clear: If students can learn mathematics with understanding, they should do so, even if the timing doesn’t fit the intended curriculum. What does this mean for these early forays into the integers?

I think there are two primary inroads: 1) Take up the issue when it arises (generally sometime in the first or second lesson on multi-digit subtraction with remainders); or 2) build it into the curriculum early and often. The first is the easier to prepare for, but trickier to respond to if one is not accustomed to discussing signed numbers with students. The second is harder, requiring preparation, but has the benefit of, well, preparation. You can anticipate students’ responses and have routines prepared for addressing their issues.

**Taking Up the Integers When They Arise Naturally**

As I mentioned earlier, I cannot recall a second-grade class where the notion of “special” numbers (negative numbers in adult parlance) that must be marked and treated differently, did not arise. Generally, my teacher colleagues tend to feel uncomfortable with taking on this issue when it occurs, because it takes time away from the intended exercises, and not all student in the class have come to the same conclusion simultaneously. But if students can count into the decades, research has shown that they can comprehend groups of 10 and groups of 1s and can count-up, and sometimes count-down to compare two different quantities. This is all that is needed for multi-digit subtraction.

This sometimes kicks in as early as kindergarten, but more generally in the first grade. It is then that our notation for multiple digits becomes important. Fifty-three isn’t 5 and 3, but 5 groups of 10 units and 3 single units. Counting-on or counting-up are the only strategies available to students until they learn notation. Even after the notation is acquired, it takes some time for students to be fluent with it in 2nd or 3rd grade (Fuson, Wearne, et al., 1997).
So, when a student invents the negative numbers, the key is to draw out the strategy that spawned that invention. John Dewey (p. 181) once wrote, “Give the pupils something to do, not something to learn; and the doing is of such a nature as to demand thinking; learning naturally results.” It is the utility of the negative number, what it does for the child, that can make it memorable for others. With Kwame, one might ask him to demonstrate his strategy for the class, allowing him to share his thinking as prototypical of what others may be contemplating. Comparing his strategy to another student, as for example, Lorna who counts up from 27 to 53 while keeping track of the ones and tens separately (e.g., “28, 29, 30, 40, 50, 51, 52, 53…26”) can show that there are two jumps of 10 units and six jumps of one unit. You might keep track of this on a number line. See Figure 3.

Figure 3. Lorna’s strategy

Lorna jumps by 10s and 1s to find 26, the difference between 53 and 27.

One could also count backwards, using a number line, from 53 to 27, like JJ. See Figure 4.

Figure 4. JJ’s strategy

JJ counts backward and keeps track of the directions of his jumps.
But other, more interesting strategies are possible. Together, we call these kinds of strategies, compensation strategies, because students add on a bit and take off a bit to make the jumps of 10s and 1s easier (for them) to count:

Amanda first jumps to a decade number. Then jumps by 10s until the decade prior to the greater number. Finally, Amanda counts by ones to get to the greater number.

In this compensation strategy, Gabriel counts by 10s to the nearest decade. He then counts back by ones.

Some of these strategies are more efficient than others. Lorna used 8 total jumps to figure out the answer, while JJ used 9 jumps and Gabriel used 7 jumps. Interestingly, the “regrouping” strategy, most frequently taught for 2-digit subtraction, is the least efficient when shown in number line form and requiring 11 steps. See Figure 7.

The learner first must regroup, represented by a jump of +10, as a 10 is traded for 10 ones. Then subtract 7 from 63 to get 56. Next, subtract 10 to make-up for the 10 added. Then, subtract two more 10s. This is not a bad strategy, but mentally, others are more logical and efficient.
**Jump, Jump, Jump!**

One of the best activities I have tried with younger students is the “Jump Jump Jump” game. Note: Players can only jump by 10s or 1s. Students solve problems individually, and then compare answers with that of a partner. By drawing the jumps on a number line, even if the initial answer is incorrect, discussion with the partner will allow the player to revise the solution. Moreover, the visual nature of the representation allows the teacher to quickly see who is using more sophisticated compensation strategies, and who is consistently thinking of different ways to make subtraction more efficient. The notes that the teacher takes may be used to guide whole group/class discussion after the game is played.

Here is the game. It is adapted from a curriculum project I worked on that utilized what we know about children’s development of multi-digit number concepts and operations to create simple models that students could use by bridging from their earlier knowledge, in this case counting, to more sophisticated understanding of the structure of our place value system.

**Materials:**

- Use a 20-sided die. These are easily found in educational supply stores and game stores.

- Number Lines, one per player: I like to make them pretty sparse so that the students can fill in the in-between numbers as they play. For example, this number line only shows the decades. For younger students you may want to fill in the 5’s (see below).

![Number Line](image)

**Rules:**

**Level 1 (addition only):**

- Two Players
- Choose a player to begin. This player rolls the die twice.
- On own number line, each player must find the sum of the two rolls using jumps of only 10 or 1. Each player keeps track of own total number of jumps.
  - You can jump forwards or backwards or some combination.
  - You can start at any number on the number line.
- Player finding the sum with the least number of jumps wins the round.
- After 5 rounds, the winner is the player who won the greater number of rounds.
Example: Adi rolled a 13 and Ku rolled a 9.

![Figure 8. A typical Level 1 round](image)

As can be seen, Ku’s strategy is more efficient. However, both are correct. Ku win’s this round, but both students can now see that 9 can be represented by +10 - 1.

**Level 2 (Addition and Subtraction):**

- Using the same rules, only making jumps of 10 or 1, determine the difference between the two numbers rolled.

Example: Adi rolled 12 and Ku rolled 18.

![Figure 9. Here Ku required 3 jumps while Adi’s used a clever, but less efficient strategy.](image)
Level 3 (Addition and Subtraction to reach 2-digit target number)

- Group in pairs. Don’t use the die this time. Each person secretly chooses a 2-digit number between 10 and 100 and records it on post-it notes. (Alternatively, the teacher can make cards with two and three-digit addition and subtraction problems that students can select from).
- Reveal the numbers.
- Find the difference between the two numbers using jumps of only 10 or 1.

Example: Adi and Ku solve the problem, $72 - 19 = \square$

![Image of number line showing Ku's and Adi's strategies](image)

Figure 10. Ku’s and Adi’s Strategies

Ku’s and Adi’s strategies became more sophisticated in different ways. Here Adi’s strategy is more efficient. Subtracting 19 is the same as subtracting 20 and then adding 1. But Ku has begun counting without having to explicitly mark his number line. This shows that he is gaining the knowledge of the decades and the regularity of the base 10 numbers (at least two-digit numbers).

One of the nice discussions I have witnessed in classrooms who use this type of activity centers on the fact that it doesn’t matter where you start. You must keep track of the jumps and their directions. The difference between any two numbers is reflected in the sum of the positive jumps minus the sum of the negative jumps. Instead of starting at 72, for example, Adi could have started at 70, jumped +2, then jumped -20, and then jumped +1.

Seventy plus 3 – 20 is the same as 72 – 19. For older students, you can symbolize this algebraically and work through the sums systematically to prove that:

\[
70 + 2 - 20 + 1 = 72 - 20 + 1 \\
73 - 20 = 72 - 19 \text{ or} \\
71 - 18 = 52 + 1 \text{ etc.} \\
53 = 53
\]
This activity provides good practice with addition and subtraction facts. As well, it builds students’ mental mathematics skills, and has the more important advantage of building number sense while emphasizing the properties of equality.

Depending on the tools you have in your classroom, and the abilities of your students to count by 10s and 1s, you can adapt these activities to utilize counters or Base 10 Blocks, or ten-frames, or different apps. The important thing to remember is to emphasize the *difference*—i.e., the distance—between the minuend and the subtrahend. Doing so will allow students to use integers meaningfully early on, so that when they get to focus on them later in the curriculum, they already have the conceptual understanding and some procedural skill for what signed numbers mean.
References


James A. Middleton, Ph.D. is Professor of Mechanical and Aerospace Engineering, and Mathematics Education at Arizona State University. He is an international expert on mathematical thinking, student motivation in mathematics, and teacher change in mathematics. Currently, he is studying student engagement in high school mathematics. At ASU, he teaches experimental statistics for mechanical and aerospace engineers and freshman design courses. He also supervises student capstone design projects. Nationally, Jim serves on the Academic Advisory Council for the College Board and is Co-Chair of the Pre-AP Mathematics committee. Some of his greatest times have been working with teachers in their classrooms, researching student thinking, and conducting professional development. PD focuses on helping teachers develop deeper content understanding of algebra, geometry, and discrete mathematics and, on some occasions, afterward at a pub, laughing and trying to put all that was learned into the larger context of being a teacher. Jim is also an accomplished musician and performer.
Great Problems Often Lead to Bigger Ideas!

Marian Small

Abstract
Some of the richest mathematics problems are those that lead to generalizations. They are rich because they are accessible to almost all students, are useful in leading to substantial computational practice for those who need it most, and most importantly, they lead to opportunities for students to apply standards of mathematical practice. Problems in this article are organized by grade bands: K-2, 3-5, and 6-8 grade.

Great Problems Often Lead to Bigger Ideas!
We often hear about the importance of engaging students with math problems that are meaningful. But there are different ways to look at what is meaningful. One way is to ensure that the problems relate to students’ experiences outside of school, focusing on topics of interest to them, and ideally topics where knowing mathematics is useful. Another way that resonates with many students is to show them how one problem fits into a bigger picture - a piece of a larger interesting puzzle that leads to generalizations. This is as relevant in Grade 1 as in Grade 8. What follows are problems in three grade bands and discussions of the features of those problems.

Grades K-2
Problem 1:
You add two numbers.
You subtract the same two numbers.
The sum is exactly 10 more than the difference.
What could be the two numbers?

There are many nice things about this problem. 1) It is accessible to all students since there are numerous solutions. For younger students, the solution may be: 10 and 5 since $10 + 5 = 15$ and $10 - 5 = 5$. 2) students may get a great deal of addition and subtraction practice while exploring possibilities. 3) What matters most, in my opinion, is that this problem is an example of a bigger idea.

Try Problem 1 before you keep reading to see if you uncover the idea.
Yes, there are an infinite number of solutions and it’s easy to see why, visually. Look at this picture.

Notice that the center point can be **any number**. If you add 5 and subtract 5, the answers are 10 apart. Many students will fully understand this visual explanation.

Algebraically (although not what a primary student would write), this is a way to say that the number \((a + 5)\) is **ALWAYS** 10 more than the number \((a - 5)\).

Some students may generalize further: If the sum of two numbers is \(n\) more than the difference between the two numbers, the first number can be any number and the second number is \(n/2\). For example, if the difference between the sum and difference is 16, the second number is 8.

**Problem 2:**
You begin with a 2-digit number.
You reverse the digits.
You subtract the lesser number from the greater number.
The result is 36.
What could be the beginning number?

As with Problem 1, Problem 2 has merit for many reasons: 1) It has many solutions. There are some simple solutions (e.g., 40 – 4.) 2) It provides the opportunity for addition and subtraction practice if an answer is not immediately apparent. 3) Its greatest **value** is that there are many (not an infinite number this time) solutions that follow a pattern.

Try the problem before continuing to read.

You may want to collect solutions from different students, post them, and ask students if they notice a pattern: 40 – 4, 51 – 15, 62 – 26, 73 – 37, 84 – 48, 95 – 59. In each case the digit values are 4 apart. Why is that the case?

The algebraic explanation: If the digits are \((a + 4)\) and \(a\), then the greater number is \(10(a + 4) + a\) and the lesser one is \(10a + (a + 4)\). When you subtract, the result is: \((10a + 40 + a - 10a - a - 4)\), which is 36. But this explanation is appropriate for a middle school student, and not a primary student.

A more visual explanation can be seen using a 100s chart.

If numbers on a 100s chart are 36 apart, that means that they are either 4 lines apart on the chart with the greater number 4 to the left of the lesser number, or 3 lines apart on the chart with the greater number 6 to the right of the lesser number. The possible value pairs are shown (color-coded) in this 100s-chart.
100s-chart
Grades 3-5
You create a shape on a geoboard.
The border of the shape touches exactly 12 pegs.
There are exactly 3 pegs inside the shape.
What is the area of the shape?

This problem, as the ones described for younger students, has many solutions and leads to a generalization. Again, students should try various shapes and practice determining the area in different circumstances. Again, the solution to this problem can lead to a generalization.

For example, look at these three shapes:

Shape 1

Shape 2

Shape 3

Shape 1 is a simple rectangle with 2 rows of 4 square units. The shape’s area is 8 square units.

Shape 2 is somewhat more complicated. Decomposing it into parts (as shown below) may help you see a rectangle with an area of 3 square units, a triangle on the right with an area of half of 3 square units, a triangle on the bottom left with area of half of 3 square units, and a triangle in the bottom center with an area of half of 4 square units. The area of Shape 2 is 8 square units as is the area of Shape 1.
Shape 3 may be decomposed into two triangles, each with the area of half of 2 square units and a rectangle with area 6 square units, for a total area, again, of 8 square units.

Shape 3

So, it seems that ANY shape with a border that touches 12 pegs with 3 pegs inside has an area of 8 square units and that happens to be true. That observation could lead to a further exploration of what happens when there are either more pegs on the border or more pegs on the inside (or both). By the way, the area is always half the number of pegs touching the border plus one less than the number of inside pegs. That is Pick’s Theorem.
One of the ways to model a 2-digit by 2-digit multiplication is to create a rectangle using base ten blocks, with one number representing the length and the other number representing the width. This model shows a 12 x 12 square and uses 11 Base 10 Blocks: one 100 flat, four 10 longs and four ones.

Problem: What multiplication could you model using exactly 18 blocks instead of 11?

This problem again leads to lots of practice, as well as to generalizations. It, too, is accessible to most students, given that there are two very simple solutions; 1) 6 rows of 3 ones, or 2) 9 rows of 2 ones. Notice that these simplest solutions are possible since a deliberate decision was made not to force 2-digit by 2-digit multiplication in the question, although the model suggests it.
Possible solutions involving two-digit numbers are: 15 \times 12, 24 \times 21, 45 \times 11, 36 \times 11, and 33 \times 12. The solution using Base 10 Blocks for determining the product, 24 \times 21, is shown below.

You may notice that there are 3 “rows” with six blocks (not all the same) in each row. That is because the sum of the digits of 24 is 6, (Notice that there are 2 larger blocks and 4 smaller ones in each row (not all smaller ones have the same value) and the sum of the digits of 21 is 3. (Notice that there are 3 rows.) It is the case that whenever a one-digit sum is 6 and the other is 3, or a one-digit sum is 9 and the other is 2, there will be 18 blocks used to represent the sum.
You go to a sale. You buy a jacket that is 40% off and shoes that are 20% off. When you go to pay for the jacket and shoes on sale, the price of each is the same. What were the prices before the sale?

To solve this problem, some students may try an example. If the jacket originally cost $100, the sale price would be $60. That means the shoes also cost $60, after the 20% off. Since you only paid 80% of the original shoe price, that means that 80% of some number must be $60, so 20% must be one-fourth as much ($15). The total original cost of the shoes would have been $75. Students might conclude that the original prices are $25 apart, and they are in this case, although this is not true, in general. Students could learn that the difference need not be $25 if the teacher proposes, for example, that students explore the problem if the original price is $50 and not $100.

Some students may use algebra. If the original jacket price is $x$, then the sale price is $0.6x$. If the original shoe price is $y$, then the sale price is $0.8y$. So, $0.6x = 0.8y$ must be true. That means that $6x = 8y$, or $y = \frac{3}{4}x$. So, the original shoe price is $\frac{3}{4}$ the original jacket price. And that explains the $75 and $100 in the example described above.

Some students may use a visual approach to make sense of the problem. The picture below models a double number line approach.

![Double Number Line](image)

Notice that the two sale prices are made to match.

Notice, as well, that each 20% interval on the Shoe price line must be $\frac{3}{4}$ the “size” of the 20% interval on the Jacket price line, since 3 sets of 20% on the top line match 4 sets of 20% on the bottom line. Since each interval on the bottom line is $\frac{3}{4}$ as long as the interval on the top line, that means that the distance to 100% on the bottom line must be $\frac{3}{4}$ the distance to 100% on the top line, which is where the 75% jacket price is. The original shoe price must be 75% of the original jacket price, regardless of the show price.
Suppose that you are using a function machine. When the input is 5, the output is 22. What could the function rule be? Think of at least five possibilities.

There are solutions to this problem that involve both linear and non-linear functions. (Note: A few non-linear examples include \(n^2 - 3\), \((n+1)^2 - 14\), and \(2n^2 - 28\).) There are many linear solutions, and these are particularly interesting in terms of the pattern that may become evident. Consider the linear functions:

\[
5n - 3 \quad 4n + 2 \quad 3n + 7 \quad 2n + 12 \quad n + 17
\]

Notice that the constant increases by 5 each time the coefficient of \(n\) decreases by 1. (Since the input is 5, one less \(n\) means 5 less, so the constant must increase by 5 to compensate.) That means that there are many more examples of linear functions, as for example, \(6n - 8\), \(7n - 13\), \(-n + 27\), and \(-2n + 32\). This generalization really brings home what is going on when expressions are evaluated.

**Summary**

The examples provided here are just a few of the many, many examples I could have used to show the power of problems that are not an end-in-and-of-themselves but lead to bigger ideas. Try these with your students.

Marian Small, a former Dean of Education at the University of New Brunswick, Ontario, Canada, gives presentations internationally about K-12 mathematics education. Her foci are on teacher questioning to highlight important mathematical concepts, strategies to include all students, and methods for developing students’ critical thinking and creativity talents. Among her publications are: *Good Questions: A Great Way to Differentiate Math Instruction*, *More Good Questions: A Great Way to Differentiate Secondary Math Instruction*, *Eyes on Math*, *Uncomplicating Fractions*, *Uncomplicating Algebra*, *Building Proportional Reasoning*, *Fun and Fundamental Math for Young Children*, *The School Leader’s Guide for Building and Sustaining Math Success*, *Math that Matters: Targeted Assessment and Feedback*, *MathUp*, and *Understanding the Math We Teach and How to Teach It: K-8*. 
How Different Sudokus Can be The Same

Richard Kalman

Abstract

So many people are addicted to solving Sudoku puzzles. But let's look at how one Sudoku can be modified into more than 1½ million apparently different puzzles — without really being different!

Sudoku is a logic puzzle based on the process of elimination. In a 9×9 square, every row, every column, and every 3×3 box must include all numbers 1 through 9. As a result, no row, column or box can include the same number twice. In addition to these three constraints — row, column, box — on a number that might be placed in a cell, a fourth constraint occurs when another number already occupies that cell. When completed, our puzzle in Figure 1 becomes Figure 2.

![Figure 1. Original puzzle](image)

![Figure 2. Completed Sudoku](image)

It is interesting that when we interchange rows, columns or blocks of 3×3 boxes, one Sudoku can appear to be a completely different Sudoku. In theory, a puzzle master could give you the same puzzle twice and make you think you are viewing two different puzzles! Let’s examine five basic transformations.

* * * * *
**ROW SWAPS:** Figure 3 shows the result of swapping the red and blue rows. We could have swapped any two or even all three complete rows, providing that they lie in the same horizontal block of three 3×3 boxes. In this case we chose the topmost block. This keeps all nine numbers in the same row, just in a different order. Note that each column and each 3×3 box still contain all nine numbers. This is one of 216 possible interchanges.

<table>
<thead>
<tr>
<th>5</th>
<th>8</th>
<th>9</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
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<tr>
<td>4</td>
<td>6</td>
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<td>9</td>
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<td>2</td>
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<td>7</td>
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<td>9</td>
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<td>4</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Figure 1: Original puzzle

<table>
<thead>
<tr>
<th>4</th>
<th>6</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
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<td></td>
<td></td>
</tr>
<tr>
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<tr>
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</tr>
<tr>
<td>5</td>
<td>7</td>
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</tr>
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<td>6</td>
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<td>1</td>
<td>4</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Figure 3: After swap of two rows

---

**COLUMN SWAPS:** In Figure 4 we swapped two complete columns, the red and the blue. As before we could have swapped any two or all three columns if they were in the same vertical block of three 3×3 boxes. This time both columns are in the center block. Again, each row, each column, and each 3×3 box will contain all nine numbers. As in Figure 3 there are 216 possible swaps.

<table>
<thead>
<tr>
<th>5</th>
<th>8</th>
<th>9</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
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<td>4</td>
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<td>9</td>
<td>3</td>
<td>5</td>
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</tr>
<tr>
<td>5</td>
<td>7</td>
<td></td>
<td></td>
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<tr>
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<td>9</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Figure 1. Original puzzle

<table>
<thead>
<tr>
<th>5</th>
<th>9</th>
<th>8</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
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<tr>
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<td>4</td>
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<td>6</td>
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<tr>
<td>9</td>
<td>3</td>
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<td>7</td>
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<tr>
<td>5</td>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>5</td>
<td>3</td>
</tr>
</tbody>
</table>

Figure 4. After swapping two columns
**HORIZONTAL BLOCK SWAPS:** Figure 5 shows the swapping of the red and blue horizontal blocks of 3×3 boxes. Once more, we can swap any two or all three horizontal blocks. As above, each box keeps its numbers and each row and column still has all nine numbers, just in another order. There are six possible ways to interchange vertical blocks.

<table>
<thead>
<tr>
<th>5</th>
<th>8</th>
<th>9</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
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<tr>
<td>6</td>
<td>7</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

Figure 1. Original puzzle

| 5 | 7 | 1 | 4 |
| 6 | 9 | 1 | 4 |
| 9 | 2 | 3 | 5 |
| 2 | 4 | 6 | 9 |
| 6 | 7 | 8 | 1 | 2 | 3 |
| 9 | 3 | 5 | 7 |

Figure 5. Switching two horizontal blocks

* * * * *

**VERTICAL BLOCK SWAPS:** In Figure 6 the red block of 3×3 boxes switches with the blue block. Again, this preserves the fact that every row, every column, and every box will contain all nine numbers. As in Figure 5, there are six possible ways to swap these blocks.

<table>
<thead>
<tr>
<th>5</th>
<th>8</th>
<th>9</th>
<th>3</th>
</tr>
</thead>
<tbody>
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<td>2</td>
<td>4</td>
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<td>7</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>6</td>
<td>9</td>
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<td>6</td>
<td>7</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

Figure 1. Original puzzle

<table>
<thead>
<tr>
<th>3</th>
<th>8</th>
<th>9</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>9</td>
<td>6</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Figure 6. Switching two vertical blocks
**DIGIT SWAPS:** Figure 7 illustrates the most obvious way to change the look of the Sudoku without changing the nature of the puzzle itself. Replace each number according to a table like the following:

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 1. Swapping numbers

We will change every number using Table 1. For example, each 5 is replaced by a 7. As we fill in the blank cells below, every new entry will obey the entries in the table. The physical layout of the Sudoku doesn’t change, but every number does. As a result, both puzzles can be solved with the exact same series of steps, so we can consider them as the “same” basic Sudoku.

<table>
<thead>
<tr>
<th>5</th>
<th>8</th>
<th>9</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>7</th>
<th>1</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>8</td>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>

2. Reflect every number over either of the two diagonals, or
3. Rotate the entire puzzle 90° in either direction

The question is: Can Figure 7 and Transformations 8 and 9 produce the same results as applying a series of swaps as shown in Figures 3 through 6? Or will they produce several new configurations of the numbers? I leave that to you to figure out!
Richard Kalman was very involved in mathematical competitions from 1953 to 2014 as a student, teacher, math team coach, problem author, lecturer, book editor, and executive director, for all levels from grade four through twelve. Nationally, he was best known for his leadership positions with the American Regions Math League and the Math Olympiads for Elementary and Middle Schools. He now lives in Florida where he lectures locally on history and musicals.
Merging Math Content with Applications: Increase Interest and Achievement

Carole Greenes, Emily Branam, and Tanner Wolfram

Abstract

As noted by leading mathematics educators and researchers, problem solving is both an end result of learning and the means through which mathematics is learned (DiMatteo & Lester, 2010; Lester, 2013; Stein, Boaler, & Silver, 2003). The merging of mathematical content with applications in problem solving activities is a natural way to develop students’ problem-solving talents (Cai, 2010). Students who reason well, and are good problem solvers have excellent observation skills, remember seemingly trivial information, generate numerous ideas, look at things in different ways, and wonder (Lester, 2013; Wood, Turner & Civil, 2016).

To further develop and improve their problem-solving skills, students must have opportunities to solve a variety of types of problems/explorations (ADE, 2017; NCTM, 2010), over extended periods of time, and work with other students in small group arrangements. During their explorations, students will:

- Explore – Investigate
- Observe – Describe
- Hypothesize – Brainstorm
- Wrestle with Ideas – Construct Explanations/Proofs/Arguments
- Collect Data, Organize and Analyze Those Data
- Use Various Technologies (e.g., phone apps for calculating, graphing, measuring, analyzing data)

The applications of mathematics in the problems that follow were designed to address the actions identified above. Below are descriptions of these applications in the order in which they appear.

**Braille Algebra** begins by describing Braille, why the Braille system was developed, and how the Braille system represents numbers. Problems require solution of algebraic equations to identify values of Braille symbols, for numbers 1 through 10.

**Company Logos**, that are common identifiers for major companies, provide the contexts for geometric and measurement problems, as well as for the application of arithmetic computations. Company logos analyzed are Microsoft, Target, Toyota, and Chevrolet. Problems require identification and analyses of perimeters and areas of the logo shapes; computation of percentages of shapes/colors within a logo; identification of lines of symmetry of shapes and decomposition of shapes to identify internal shapes; and application of formulas/functions for computing and analyzing circumferences of circles and ovals.
Gold connects mathematics with art and history. Gold Jewelry requires identification of percent of gold in jewelry items based on ratios showing amount of gold and other metals as related to 24k gold, and completion of a table of data. Golden Ring involves interpretation of the chart completed in Gold Jewelry, and comparison of amounts of gold by weight. In Gold Story, the story is completed by inserting numbers provided. The location of each number in the story is based on analyses of the mathematical relationships, magnitudes and types of numbers given. Gold Rush requires interpretation and computation of inflation rates of objects from the time of the Gold Rush to the present.

Letter Frequency: Most Popular Letters involves analyses of different type of printed materials to identify most frequently appearing letters of the alphabet. Does that frequency vary by the type of text? Type of letter (e.g., vowels, letters used to represent quantities)? Working in groups, students select and examine four different print materials. Letter frequency is tabulated and analyzed for each, and for the total.

Phone and Tablet Teasers requires calculations of proportions based on heights and widths of screens of different devices. Graphs of screen sizes enable connections between proportions and slopes of lines.

Quakes involves the interpretation of magnitudes of earthquakes using the Richter Scale. In Problem 1, the percent of earthquakes occurring in different U.S. states since 2015 are computed. Problems 2 and 3 are good examples of the use of logarithms – a quantity representing the power to which a fixed number (the base – in these problems, base 10) must be raised to produce a given number. Exponents are decimal fractions. With a calculator, values of the exponents can be computed directly.

Unusual Measurement Units introduces five measurement units, three from ancient times and two more recent. All problems require data collection and analyses. Four problems require length measurements. The fifth requires conversions among minutes, days and years.
Braille Algebra

Braille is a system to enable people who are blind or who have low vision, to read. Each Braille symbol has a specific arrangement of raised dots in a 6-space grid. The inventor of Braille, Louis Braille, went blind at the age of 3. At the age of 15, he developed the 6-dot Braille system, based on an earlier 12-dot system. Braille numbers use this four-dot symbol to the left of each number to indicate that what follows are numbers. (Note: Numbers are not in order.)

Symbol preceding the number

<table>
<thead>
<tr>
<th>Variable</th>
<th>Braille Symbol</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>⠁</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>⠼⠉</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>⠼⠁</td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>⠼⠓</td>
<td></td>
</tr>
<tr>
<td>e</td>
<td>⠼⠑</td>
<td></td>
</tr>
<tr>
<td>f</td>
<td>⠼⠊</td>
<td></td>
</tr>
<tr>
<td>g</td>
<td>⠼⠃</td>
<td></td>
</tr>
<tr>
<td>h</td>
<td>⠼⠋</td>
<td></td>
</tr>
<tr>
<td>i</td>
<td>⠼⠙</td>
<td></td>
</tr>
<tr>
<td>j</td>
<td>⠼⠚</td>
<td></td>
</tr>
<tr>
<td></td>
<td>⠼⠛</td>
<td></td>
</tr>
</tbody>
</table>
Solve the following equations to identify the Braille numbers, 1-9. Each variable corresponds to a different Braille number. Fill in the value of each Braille symbol in the chart above.

\[ 4(a^2 + a - 1) = 8 + 4a^2 \]
\[ 14 + 12b^2 = 26 \]
\[ \frac{c}{2} - 6 = 6 - c \]
\[ 4 + \frac{d}{5} = 5 \]
\[ 10 + e = \frac{4e}{3} + 7 \]
\[ 10 = \frac{9}{2} f + 1 \]
\[ 7g + 6 = 6(2 + g) \]
\[ \frac{3}{4} \sqrt{h} - 4 = \frac{-10}{4} \]
\[ \frac{3}{2} i + \frac{3}{4} = \frac{3}{4} + i \]
\[ \frac{j + 8}{3} = 5 \]
### Braille Algebra Solutions:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Braille Symbol</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
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<td>3</td>
</tr>
<tr>
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</tr>
<tr>
<td>c</td>
<td>⠼⠓</td>
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</tr>
<tr>
<td>d</td>
<td>⠼⠑</td>
<td>5</td>
</tr>
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<td>g</td>
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</tr>
<tr>
<td>j</td>
<td>⠼⠛</td>
<td>7</td>
</tr>
</tbody>
</table>
Company Logos

A logo is a design used by a company to identify its products or vehicles. Below are four well-known logos, all of which are VERY mathematical! Note that all logos pictured below are not the sizes of the actual logos. When these pictures of logos below are printed, their measurements and colors may be slightly different from what is presented here.

Microsoft Logo

Microsoft was founded in 1975 by Bill Gates and Paul Allen while working in Gates’ garage in Albuquerque New Mexico. Today, Microsoft’s headquarters is in Redmond, Washington. Microsoft sells computer software, computers, and other technology. Below is the Microsoft logo. The colors of the square are orange, green, blue, and yellow, with white separating the colors.

For the problems that follow, please use the bordered Microsoft logo shown below. (Note: The black border is included only to assist visualization and should not be considered in questions that require measuring.)

Each side of the blue square (bottom left square) is 0.5 inch. The length from the bottom left corner of the blue square to the bottom right corner of the yellow square is 1.125 inches. The four colored squares are congruent.

1. How many lines of symmetry does the Microsoft logo have? (Do not include color in your determination.)

2. What is the area of the Microsoft logo?

3. What is the perimeter of the Microsoft logo?

4. What percent of the logo is covered by colored squares? Describe how you solved this problem.
Target Logo

Target is a general merchandise store that is present in all 50 states in the U.S. with headquarters in Minneapolis, Minnesota. Before it was named Target in 1962, it was known as The Dayton Dry Goods Company and named after its founder, George Dayton.

In this picture, the diameter of the full circle of the Target logo is 2 inches. The diameter of the small red circle is 0.7 inch. The thickness of the white band is 0.3 inch. For these problems, use $\pi = 3.14$. Give all solutions to two decimal places.

1. What is the circumference of the Target logo?

2. How many times greater is the radius of the large outer red circle than the radius of the inner red circle?

3. What is the area of the Target logo?

4. What percent of the Target logo is white? What percent is red?
Toyota Logo

Toyota is a Japanese car company that first began selling cars in the United States in 1957.\textsuperscript{1} Toyota’s headquarters is in Aichi, Japan.

Use the figure below to solve problems 1, 2, and 3.

1. How many ovals are in the Toyota Logo?

2. In the table below are distances between two points on the logo above. Determine the circumferences of each of the ovals\textsuperscript{3}. Use 3.14 for π. Use the formula for circumference of an oval: $2\pi \sqrt{\frac{\text{semi major axis}^2 + \text{semi minor axis}^2}{2}}$

<table>
<thead>
<tr>
<th>Points</th>
<th>Distance (centimeters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a to b</td>
<td>5</td>
</tr>
<tr>
<td>h to k</td>
<td>2</td>
</tr>
<tr>
<td>c to d</td>
<td>3.5</td>
</tr>
<tr>
<td>i to j</td>
<td>1.5</td>
</tr>
<tr>
<td>e to f</td>
<td>6.5</td>
</tr>
<tr>
<td>h to g</td>
<td>4.5</td>
</tr>
</tbody>
</table>

3. How many times greater is the circumference of the largest oval than the circumference of the smallest oval?
Chevrolet Logo

Chevrolet is one of the largest car manufacturing companies in the United States. It is owned by General Motors with headquarters in Detroit, Michigan.

For the questions that follow, consider only the gold portion of the Chevrolet logo. Use the points labeled in the picture.

1. How many parallelograms can you form by connecting the labeled points on the Chevrolet logo? (Note: Sides of parallelograms may contain more than two points.)

2. How many triangles can be formed by connecting labeled points on the Chevrolet logo? Triangles can only be formed if connecting lines do not cover more than one labeled point (e.g. a triangle kje is not counted since the line connecting j and e passes through points i and f). And, two sides of each triangle must be on the border edge of the logo.
3. The Chevrolet Mall Store sells three sizes of the Chevrolet logo\(^2\). The figure below gives information about the Small Chevrolet logo.

On the Small Chevrolet logo with labeled points, if the distance from \(b\) to \(c\) is 1.5 inches, the distance from \(c\) to \(d\) is 7 inches, the distance from \(f\) to \(e\) is 5.5 inches, what are the area and perimeter of the logo?
Solutions
Microsoft Logo:
1. Four
2. \(1.125 \times 1.125 = 1.266\) The area of the bordered logo is 1.266 square inches.
3. \(1.125 \times 4 = 4.5\) inches The perimeter of the bordered logo is 4.5 inches.
4. Since the four small squares in the Microsoft logo are congruent, the area of each square, including the blue square, is \(0.5 \times 0.5\), or 0.25 square inches. The total area of the four colored squares is \(4 \times 0.25\), or 1 square inch. From Question 2, the area of the bordered Microsoft logo is 1.266 square inches. Then: \(1 / 1.266 \times 100 = 79.0\). So, 79.0% of the bordered logo is covered by colored squares.

Target Logo:
1. \(D = 2 \pi\) Then: \(2\pi = 6.28\) inches. Circumference of the Target logo is 6.28 inches.
2. \(1/0.35 = 2.86\) The radius of the large red circle is 2.86 times greater than the radius of the small red circle.
3. Area of the Target logo is \(3.14 \times 1^2\), or 3.14 square inches.
4. Area of the Target logo is 3.14 square inches.
   Area of the white circle (with no inner red circle) is \(\pi \times 0.65^2\), or 1.33 square inches.
   Area of inner red circle is \(\pi \times 0.35^2\), or 0.38 square inch.
   Area of white band is \(1.33 – 0.38\), or 0.95 square inch.
   \(0.95/3.14 \times 100 = 30.3\). So, 30.3% of the Target logo is white. So, \(100 – 30.3\), or 69.7% is red.

Toyota Logo:
1. 3
2. Oval with the points ahbk: Semi major axis is \(5/2 = 2.5\). Semi minor axis is \(2/2 = 1\)
   Using the formula, the circumference is 12.0 centimeters.

   Oval with the points cjdi: Semi major axis is \(3.5/2 = 1.75\). Semi minor axis is \(1.5/2 = 0.75\)
   Using the formula, the circumference is 8.46 centimeters.

   Oval with the points ehfg: Semi major axis is \(6.5/2 = 3.25\). Semi minor axis is \(4.5/2 = 2.25\)
   Using the formula, the circumference is 17.6 centimeters.

3. The Circumference of the largest oval is 17.6 cm. The circumference of the smallest oval is 8.46. \(17.6/8.46 = 2.08\). The circumference of the largest oval is 2.08 times greater than the circumference of the smallest oval.
Chevrolet Logo:
1. Seven: kdej, abcl, ifgh, abfi, lcgh, lcfi, abgh
2. Eight: cde, def, jkl, kij, ihg, fhg, abc, abl
3. Perimeter = 53.54 inches  
   Area = 99 square inches

More detailed explanation for #3 (Perimeter)
Givens:
- **b to c; a to l; f to g; and i to h** are each 1.5 inches long.
- **c to d and j to i** are each 7 inches long.
- **f to e and k to l** are each 5.5 inches long.

Solve for distance from **a to b (and h to g):**
- **c to d** is 7 inches.
- The distance from **k to l** is 5.5 inches.
- Then, the distance from **a to b (and h to g)** is $19.5 - 7 - 5.5$, or 7 inches long.

Solve for the distance from **j to k (or d to e):**
- The distance from **j to k** can be found by using the formula for computing the length of the hypotenuse of a right triangle.
- Because **a to l and i to h** are both 1.5 inches long, and because the distance from **a to h** is 7 inches, the distance from **l to i** (which is also the height of this right triangle) is $7 - 1.5 - 1.5$, or 4 inches long.
- The base of the right triangle is the difference between the length of **j to i and k to l.**
  - **j to i** is 7 inches. **k to l** is 5.5 inches. The base of this right triangle is $7 - 5.5 = 1.5$ inches long.
- Apply the Pythagorean Theorem: **j to k (and d to e)** is 4.27 inches.
More detailed explanation for #3 (Area).
Determining the perimeter of the logo will provide all dimensions needed to calculate the area of
the Chevrolet logo.

Areas:
- Rectangles **abcd** and **ifgh** are each 1.5 x 7, or 10.5 square inches.
- Rectangle **lfci** is 4 x 7, or 28 square inches.

Now we need to solve for the area of either the shape **jkli** or **cdef**: (We will analyze **jkli**)
- Split the area of the shape **jkli** into a rectangle and a right triangle. The rectangle is shown in the figure blow:

![Rectangle and Right Triangle](image)

- The distance from **l** to **i** (solved in the perimeter portion of the problem) is 4 inches.
- The distance from **k** to **l** is 5.5 inches.
- The area of the rectangle is 4 x 5.5 = 22 square inches.
- The dimensions of the right triangle with hypotenuse **j** to **k** were determined in the perimeter portion of this problem. Base = 1.5, Height = 4.
- Area of the right triangle with the hypotenuse **j** to **k** is 3.
- The area of the shape **jkli** (or **cdef**) is: 22 + 3 = 25

Thus, the total area of the Chevrolet logo is 2 x 10.5 + 28 + 2 x 25 = 99 square inches.

Useful References:
**Gold**

**Gold Jewelry**

Because gold is too soft and malleable, gold jewelry is typically made with a mixture of gold and other metals. The chart below shows the number of karats of gold in typical gold jewelry, and the amount of gold in the mixture.

1. Complete the table to show the Parts of Gold as Percent of Gold

<table>
<thead>
<tr>
<th>Karat Gold</th>
<th>Parts Gold</th>
<th>% Gold</th>
</tr>
</thead>
<tbody>
<tr>
<td>9k</td>
<td>9/24</td>
<td></td>
</tr>
<tr>
<td>10k</td>
<td>10/24</td>
<td></td>
</tr>
<tr>
<td>12k</td>
<td>12/24</td>
<td></td>
</tr>
<tr>
<td>14k</td>
<td>14/24</td>
<td></td>
</tr>
<tr>
<td>18k</td>
<td>18/24</td>
<td></td>
</tr>
<tr>
<td>22k</td>
<td>22/24</td>
<td></td>
</tr>
<tr>
<td>24k</td>
<td>24/24</td>
<td></td>
</tr>
</tbody>
</table>

2. Use the table to estimate the number of karats of gold that are in each piece of jewelry shown below. (For this problem, assume all metals have the same number of atoms per gram).

<table>
<thead>
<tr>
<th>Necklace</th>
<th>Watch</th>
<th>Earrings</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Necklace" /></td>
<td><img src="image" alt="Watch" /></td>
<td><img src="image" alt="Earrings" /></td>
</tr>
<tr>
<td>4.7 grams gold, 1.567 grams copper = _________ karats</td>
<td>20 grams gold, 4.76 grams zinc, 9.53 grams silver = _________ karats</td>
<td>0.25 grams gold, 0.0172 grams platinum, 0.0057 grams copper = _________ karats</td>
</tr>
</tbody>
</table>
Gold Ring

The actual inside diameter of this 12k gold ring is 18 mm. The outside diameter is 20 mm. The ring is 4 mm thick. 1 cubic mm of gold weighs 0.019 g. (Assume all atoms in the ring have the same volume).

How many more grams of gold would be in this ring if it were made of 18k rather than 12k gold? Use the chart below.

<table>
<thead>
<tr>
<th>Karat Gold</th>
<th>Parts Gold</th>
<th>% Gold</th>
</tr>
</thead>
<tbody>
<tr>
<td>9k</td>
<td>9/24</td>
<td>37.5</td>
</tr>
<tr>
<td>10k</td>
<td>10/24</td>
<td>41.67</td>
</tr>
<tr>
<td>12k</td>
<td>12/24</td>
<td>50</td>
</tr>
<tr>
<td>14k</td>
<td>14/24</td>
<td>58.33</td>
</tr>
<tr>
<td>18k</td>
<td>18/24</td>
<td>75</td>
</tr>
<tr>
<td>22k</td>
<td>22/24</td>
<td>91.67</td>
</tr>
<tr>
<td>24k</td>
<td>24/24</td>
<td>100</td>
</tr>
</tbody>
</table>

Gold Story
Use the numbers in the rectangle to complete the story.

The largest solid gold nugget ever found weighed _________ kg and was nicknamed the Welcome Stranger. The world’s largest solid-gold statue is the Golden Buddha in Bangkok, Thailand. The Golden Buddha weighs _________ kg. It would take the amount of gold in _________ gold nuggets the size of Welcome Stranger to make the Golden Buddha. A British jewelry maker made an iPad with _________ kg of gold. You could make _________ gold iPads using the gold in Welcome Stranger.
Gold Rush

During the California Gold Rush, merchants discovered they could sell supplies to miners for extremely high prices. A shovel selling for about $36 during the Gold Rush would cost $1,000 today!

1. Complete the table using the same inflation rate to determine the cost of each item today. Round costs to the nearest dollar.

<table>
<thead>
<tr>
<th>Item</th>
<th>Gold Rush Cost</th>
<th>Equivalent Cost Today</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shovel</td>
<td>$36</td>
<td>$1,000</td>
</tr>
<tr>
<td>Cheese</td>
<td>$25 per pound</td>
<td></td>
</tr>
<tr>
<td>Beef</td>
<td>$10 per pound</td>
<td></td>
</tr>
<tr>
<td>Flour</td>
<td>$13 per bag</td>
<td></td>
</tr>
</tbody>
</table>

2. During the Gold Rush, how much would it cost a miner to make a cheeseburger which used ¼ pound of beef and 0.8 ounces of cheese? Assume all other ingredients have no additional cost. __________

3. What would be the cost of that cheeseburger today? __________
Gold Solutions:

Gold Jewelry
1. 

<table>
<thead>
<tr>
<th>Karat Gold</th>
<th>Parts Gold</th>
<th>% Gold</th>
</tr>
</thead>
<tbody>
<tr>
<td>9k</td>
<td>9/24</td>
<td>37.5</td>
</tr>
<tr>
<td>10k</td>
<td>10/24</td>
<td>41.67</td>
</tr>
<tr>
<td>12k</td>
<td>12/24</td>
<td>50</td>
</tr>
<tr>
<td>14k</td>
<td>14/24</td>
<td>58.33</td>
</tr>
<tr>
<td>18k</td>
<td>18/24</td>
<td>75</td>
</tr>
<tr>
<td>22k</td>
<td>22/24</td>
<td>91.67</td>
</tr>
<tr>
<td>24k</td>
<td>24/24</td>
<td>100</td>
</tr>
</tbody>
</table>

2. 
   a. Necklace = 18 karats
   b. Watch = 14 karats
   a. Earrings = 22 karats

Gold Ring
12k gold:
\[ V = ((\pi x 10^2) - (\pi x 9^2)) * 4 = 238.76 \text{ mm}^3 \]
238.76 mm\(^3\) x 50% = 119.38 mm\(^3\) gold
119.38 mm\(^3\) x 0.019 g = 2.27 g

18k gold:
238.76 mm\(^3\) x 75% = 179.07 mm\(^3\)
179.07 mm\(^3\) x 0.019 g = 3.40 g
3.40 – 2.27 = 1.13 g more gold

Gold Story
The largest solid gold nugget ever found weighed 72.04 kg and was nicknamed the *Welcome Stranger*. The world’s largest solid-gold statue is the Golden Buddha in Bangkok, Thailand which weighs 5500 kg. It would take the amount of gold in 76.35 gold nuggets the size of *Welcome Stranger* to make the Golden Buddha. A British jewelry maker made an iPad with 2 kg of gold. You could make 36 gold iPads using the gold in *Welcome Stranger*. 
Gold Rush

1. $10/lb x 0.25 lb = $2.50 for ¼ lb beef
   $25/lb x 0.05 lb = $1.25 for 0.8 oz cheese
   $2.50 + $1.25 = $3.75

2. The cost of a cheeseburger during the gold rush increased by the inflation rate:
   $3.75 x (1000/36) = $104.17

<table>
<thead>
<tr>
<th>Item</th>
<th>Gold Rush Cost</th>
<th>Equivalent Cost Today</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shovel</td>
<td>$36</td>
<td>$1,000</td>
</tr>
<tr>
<td>Cheese</td>
<td>$25 per pound</td>
<td>$694</td>
</tr>
<tr>
<td>Beef</td>
<td>$10 per pound</td>
<td>$278</td>
</tr>
<tr>
<td>Flour</td>
<td>$13 per bag</td>
<td>$361</td>
</tr>
</tbody>
</table>
Letter Frequency

Which letter in the English alphabet is used most frequently in printed materials? Does that frequency vary based on the type of print material, as for example, math books, novels, history books, magazines, news articles, prayer books, …? Predict first. Then check it out.

Directions:

- Pick one collaborator.
- Together, choose six different print materials.
- For each type of print material, predict the most “popular” letter.
- On the frequency table, record each type of print material.
- Predict, for each of the six, which letter will appear most frequently. Record your prediction.
- Then: for each of the print materials:
  1) Identify a passage consisting of 100 consecutive words. (Note: Consider an equation/function/formula as one word)
  2) Count the number of letters only in that passage.
  3) Tabulate the number of each of the letters, a through z, in the Letter Frequency Table.

Analyses:

1. In each print material, which letter appears most frequently? Least frequently? Were your predictions right-on?
2. In all six print materials, in total, which letter appears most frequently? Least frequently?
3. What percent of the total number of letters in each print material are each of the following?
   a. vowels  b. the letter x  c. the letter y  d. the letter z
4. What percent of all words in all six print materials begin with the letter a?
**Letter Frequency Table:**

<table>
<thead>
<tr>
<th>Letter</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td></td>
</tr>
<tr>
<td>G</td>
<td></td>
</tr>
<tr>
<td>H</td>
<td></td>
</tr>
<tr>
<td>I</td>
<td></td>
</tr>
<tr>
<td>J</td>
<td></td>
</tr>
<tr>
<td>K</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
</tr>
<tr>
<td>P</td>
<td></td>
</tr>
<tr>
<td>Q</td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
</tr>
<tr>
<td>S</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td></td>
</tr>
<tr>
<td>U</td>
<td></td>
</tr>
<tr>
<td>V</td>
<td></td>
</tr>
<tr>
<td>W</td>
<td></td>
</tr>
<tr>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td></td>
</tr>
<tr>
<td>Z</td>
<td></td>
</tr>
<tr>
<td>Total Number of Letters</td>
<td></td>
</tr>
</tbody>
</table>
Some history:

René Descartes (1596–1650) was a mathematician, scientist and philosopher. There is a story that when his works were printed, he was the first to use $x$ to represent an unknown quantity. The story: In the “old days”, books were printed using a process of typesetting, where blocks, each with a different letter, were arranged in order to produce words on pages of manuscripts. Once the blocks of letters were arranged, paint and paper were placed on top of the letters and pressed to produce a page of text. When Descartes met with the book printer to produce his famous book, *La Géométrie*, the printer suggested that $x$ be the letter used for the unknowns since few words used $x$ and he had lots of $x$ blocks. Descartes also used $y$ and $z$ to represent unknowns.
Phone and Tablet Teasers

Screen sizes of items like laptops, tablets, and phones are measured using the diagonal. Devices are then described using both the device name and the diagonal length. Tablet and phone sizes vary, so one 7” tablet may not be the same dimensions as another 7” tablet.

1. Complete the table to show the screen proportion (height/width) for each device.

<table>
<thead>
<tr>
<th>Device - Diagonal</th>
<th>Height (in.)</th>
<th>Width (in.)</th>
<th>Screen Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Galaxy Note 2 - 5.5”</td>
<td>4.80</td>
<td>2.70</td>
<td></td>
</tr>
<tr>
<td>iPhone XS - 5.8”</td>
<td>5.25</td>
<td>2.37</td>
<td></td>
</tr>
<tr>
<td>Nexus 7 - 7”</td>
<td>5.94</td>
<td>3.71</td>
<td></td>
</tr>
<tr>
<td>iPad Mini - 7.9”</td>
<td>6.32</td>
<td>4.74</td>
<td></td>
</tr>
<tr>
<td>Galaxy Note 8 - 6.3”</td>
<td>5.34</td>
<td>3.33</td>
<td></td>
</tr>
<tr>
<td>iPad - 9.7”</td>
<td>7.76</td>
<td>5.83</td>
<td></td>
</tr>
<tr>
<td>Galaxy Tab 2 - 10.1”</td>
<td>8.57</td>
<td>5.35</td>
<td></td>
</tr>
</tbody>
</table>

2. Use the table to answer these questions.
   a. Which devices have the same screen H/W ratios? __________________________

   b. If you plot sizes of devices with the same screen ratios, width on the x axis and height on the y axis, what do you predict the slope of that line will be? Why?
      ______________________________________________________________________

3. Plot sizes of the devices with the same screen ratios on the attached graph. Indicate the slope of each line on the graph. You can use either the graph provided, or use an online graphing software app (e.g., Desmos.com).
Phone and Table Teasers Solutions:

1. | Device - Diagonal | Height (in.) | Width (in.) | Screen Ratio |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Galaxy Note 2 - 5.5”</td>
<td>4.80</td>
<td>2.70</td>
<td>1.78</td>
</tr>
<tr>
<td>iPhone XS - 5.8”</td>
<td>5.25</td>
<td>2.37</td>
<td>2.22</td>
</tr>
<tr>
<td>Nexus 7 - 7”</td>
<td>5.94</td>
<td>3.71</td>
<td>1.60</td>
</tr>
<tr>
<td>iPad Mini - 7.9”</td>
<td>6.32</td>
<td>4.74</td>
<td>1.33</td>
</tr>
<tr>
<td>Galaxy Note 8 - 6.3”</td>
<td>5.34</td>
<td>3.33</td>
<td>1.60</td>
</tr>
<tr>
<td>iPad - 9.7”</td>
<td>7.76</td>
<td>5.83</td>
<td>1.33</td>
</tr>
<tr>
<td>Galaxy Tab 2 - 10.1”</td>
<td>8.57</td>
<td>5.35</td>
<td>1.60</td>
</tr>
</tbody>
</table>

2.
   a. Nexus 7, Galaxy Note 8, and Galaxy Tab 2 have the same screen ratio.
      iPad Mini and iPad have the same screen ratio.
   b. The slope would be the same as the screen ratio for each group of devices.
The first method for measuring the magnitude or strength of an earthquake is the Richter Magnitude Scale established in 1935. It is named after Charles Richter, a physicist and seismologist who worked at the California Institute of Technology in Pasadena, California. Earthquakes are rated from 1 to 10 on the Richter Scale. Each unit rating is 10 times greater than the previous unit. For example, comparing a magnitude 9 earthquake with a magnitude 8 earthquake, since $9 - 8 = 1$, the magnitude 9 quake is $10^1$ or 10 times greater than a magnitude 8 earthquake. Likewise, since $9 - 7 = 2$, a magnitude 9 earthquake is $10^2$ or 100 times greater than a magnitude 7 earthquake. The largest earthquake in the United States occurred on March 28, 1964, when a 9.2 earthquake shook Prince William Sound in Alaska.

### Earthquakes in the US Since 2015

<table>
<thead>
<tr>
<th>Date</th>
<th>Location</th>
<th>Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>May 29, 2015</td>
<td>Alaska</td>
<td>6.8</td>
</tr>
<tr>
<td>July 27, 2015</td>
<td>Alaska</td>
<td>6.9</td>
</tr>
<tr>
<td>July 29, 2015</td>
<td>Alaska</td>
<td>6.4</td>
</tr>
<tr>
<td>January 24, 2016</td>
<td>Alaska</td>
<td>7.1</td>
</tr>
<tr>
<td>September 3, 2016</td>
<td>Oklahoma</td>
<td>5.8</td>
</tr>
<tr>
<td>December 8, 2016</td>
<td>California</td>
<td>6.6</td>
</tr>
<tr>
<td>January 23, 2018</td>
<td>Alaska</td>
<td>7.9</td>
</tr>
<tr>
<td>May 4, 2018</td>
<td>Hawaii</td>
<td>6.9</td>
</tr>
<tr>
<td>August 12, 2018</td>
<td>Alaska</td>
<td>6.4 and 6.0</td>
</tr>
<tr>
<td>November 30, 2018</td>
<td>Alaska</td>
<td>7.1</td>
</tr>
<tr>
<td>July 4, 2019</td>
<td>California</td>
<td>6.4</td>
</tr>
<tr>
<td>July 5, 2019</td>
<td>California</td>
<td>7.1</td>
</tr>
<tr>
<td>August 29, 2019</td>
<td>Oregon</td>
<td>6.3</td>
</tr>
<tr>
<td>January 23, 2020</td>
<td>Alaska</td>
<td>6.2</td>
</tr>
<tr>
<td>March 18, 2020</td>
<td>Utah</td>
<td>5.7</td>
</tr>
<tr>
<td>March 31, 2020</td>
<td>Idaho</td>
<td>6.5</td>
</tr>
<tr>
<td>May 15, 2020</td>
<td>Nevada</td>
<td>6.5</td>
</tr>
</tbody>
</table>
Solve. Show solutions to two decimal places.

1. What percent of the earthquakes in the U.S. since 2015 occurred in
   a. Alaska? ________
   b. California? ________
   c. Nevada? _______

2. How many times greater is the largest earthquake in the U.S. than the largest earthquake since 2015 in
   a. Alaska? ________
   b. California? ________
   c. Nevada? ______

3. The world's largest earthquake (recorded with the Richter Scale) occurred on May 22, 1960 when a 9.5 earthquake shook Chile. How many times greater was this world’s largest earthquake than the largest earthquake in the United States? ______
Quakes Solutions:

1. Total of 17 earthquakes. 8 in Alaska. 3 in California. 1 in Nevada.
   a. Alaska $\frac{8}{17} = 47.06\%$
   b. California $\frac{3}{17} = 17.65\%$
   c. Nevada $\frac{1}{17} = 5.88\%$

2. How many times greater is the largest earthquake in the U.S. than the largest earthquake in:
   a. Alaska? $9.2 - 7.9 = 1.3 \text{ and } 10^{1.3} = 19.95 \text{ times greater.}$
   b. California? $9.2 - 7.1 = 2.1 \text{ and } 10^{2.1} = 125.89 \text{ times greater.}$
   c. Nevada? $9.2 - 6.5 = 2.7 \text{ and } 10^{2.7} = 501.19 \text{ times greater.}$

3. How many times greater was this world’s largest earthquake than the largest earthquake in the United States? $9.5 - 9.2 = 0.3 \text{ and } 10^{0.3} = (1.99 \text{ or } 2 \text{ times greater.}$
**Unusual Measurement Units**

Six unusual units of measure are described below. To gain greater understanding of these units and what they measure, conduct the experiments described below. For these, you will need to work with a partner. And, you will need some measuring tools. With some exceptions, answers vary based on the nature of the data collected.

**Span**

Since ancient times, the **handspan** or **span** has been used to measure lengths. What is the length of your handspan? Open one hand and extend and separate your fingers. **The distance from the tip of your thumb to the tip of your little finger is one handspan.**

![Handspan illustration]

Use the ruler or measuring tape to determine the length of your handspan in inches.

1. Measure the objects below in number of handspans, and with a measuring tape. Record both measurements in inches in the table below.
2. Partner completes the same table for partner’s handspans.
3. How close are the handspan lengths to measured lengths?

<table>
<thead>
<tr>
<th>Object</th>
<th>Number of Handspans</th>
<th>Total Length of Handspans</th>
<th>Measured Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width of Door</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Height of Chair Leg</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Width of Sink</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Height of Bed from Floor</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. What are your heights in spans?
Cubit

A cubit is another ancient and medieval unit of measure. A cubit is the length of your outstretched arm from your elbow to the tip of your middle finger.

![Cubit Image]

Stretch one arm. Have your partner measure the length in inches. Do the same for your partner.

1. How long is your cubit? Partner’s cubit?
2. In ancient times, the cubit ranged in length from 18 inches to 21 inches. How does the length of your cubit compare with the lengths during ancient times? Less? Greater? By what percent? Same questions for partner.
3. How many inches longer is your cubit than your handspan? What is the percent difference? Same questions for partner.

Wingspan

Wingspan or armspan is the length from one end of a person’s middle fingertip to the end of that person’s other middle fingertip, when arms are raised, outstretched, and parallel to the ground.

Stretch your arms. Have your partner measure your wingspan in inches. Do the same for your partner.

1. How long is your wingspan? Partner’s wingspan?
2. Compare your heights to wingspans? Are they the same? If not, which is greater?
3. How many inches longer is your wingspan than your span? Same questions for partner.
Pace is also a unit of length. It is either the length of:
1) One Walking Step
or
2) One Double Step (returning to the same foot)

Select a hallway or room that you can walk without obstacles. Measure the length of the room. Have your partner count your walking steps from one end of the hall/room to the other. Then count your partner’s walking steps.

1. Complete the chart.

<table>
<thead>
<tr>
<th>Partner Name</th>
<th>Room/Hall Measured Length</th>
<th>Number of Walking Steps</th>
<th>Number of Double Steps</th>
<th>Length of Room/Hall based on Walking and Double Steps</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

2. What is the length of your walking step?

3. How close is the total length of the room/hall based on number of walking steps to the measured length of your walking step? Percent difference?
Smoot: A not so old unit of measure!

One Smoot is 5’7” – the height of Oliver Smoot. To pledge a fraternity in 1958 at the Massachusetts Institute of Technology (MIT) in Cambridge, Massachusetts, Oliver had to lie down on the Harvard Bridge (connecting Boston to Cambridge across the Charles River) and move continuously (on his back) across the bridge while his fraternity brothers marked the position of his feet. The distance from each marking- head to foot (with white paint) is 5’7”. At that time, the distance across the bridge (it has changed with renovations) was “364.4 Smoots plus or minus an ear”.

Of great interest: When the bridge was repaved, the Smoots were marked again! And, Smoot later became President of the International Organization for Standardization.

1. What is your height (length) in feet and inches? Partner’s height (length) in feet and inches?
2. What is your height in Smoots? Partner’s height in Smoots?
3. The length of the Harvard Bridge today is 2,164.8 feet. How many of your body lengths are equal in length to the Harvard Bridge? How many of your partner’s lengths are equal in length to the Harvard Bridge?
Warhol

A **Warhol is 15 minutes.** It comes from Andy Warhol’s statement that “everyone will be world-famous for 15 minutes.” Warhol was a famous artist, who gained popularity for exploring popular culture in his extensive art productions, from illustrations in fashion magazines to his famous print of Campbell’s tomato soup, his favorite soup!!!!!

1. If someone is famous for 15,000 minutes or a **kiloWarhol**, how many days is that person famous?
2. If someone is famous for 15,000,000 minutes or a **megaWarhol**:
   a. How many days is that person famous?
   b. How many years is that person famous?
3. Partners: Based on your age and 15 minutes of fame/day:
   a. How many minutes have you been famous in 24 hours?
   b. How many years is that?

**Warhol Solutions:**
1. 10.42 days  
2. a. 10,416.7 days  
   b. 28.5 years  
3. Answers will vary
References


Carole Greenes, Ed.D. is Professor Emerita, Mathematics Education in the Mary Lou Fulton Teachers College at Arizona State University. Prior to that, she was Director of the Practice Research and Innovation in Mathematics Education (PRIME) Center, and Professor of Mathematics Education in the Ira A. Fulton Schools of Engineering and the College of Liberal Arts and Sciences at Arizona State University. Currently, she directs the PRIME Group that develops books of challenge problems for students, grades K – 12, and conducts research projects (Preservice Teachers Knowledge of Mathematics and Middle Students’ Algebraic Reasoning Talents.) Carole is author of more than 350 books for PreK-16 and teachers; 79 articles; six mathematical musicals; and two histories of mathematics in story and song. She is editor of the Arizona Association of Teachers of Mathematics journal, OnCore, and author of the online monthly free MATHgazine Senior (grades 8-12), MATHgazine Junior (grades 4-8), MATHgazine Elementary (grades 2-5) and MATHgazine Primary (grades K-2). In 2003, Greenes was inducted into the Massachusetts Mathematics Educators’ Hall of Fame. In 2011, she received the NCSM Ross Taylor/Glenn Gilbert National Leadership Award in Mathematics Education. In 2016, she received the Copper Apple Award for Leadership in Mathematics in Arizona, and in 2018 she received the National Council of Teachers of Mathematics Lifetime Achievement Award.

Emily Branan is a senior in the Mary Lou Fulton Teachers College at Arizona State University, studying Elementary Education with a STEM focus. She is a Project Assistant in the PRIME Group at ASU, a co-author of four articles, a co-editor of the AATM OnCore Journal, and a co-author of the MATHgazine Primary for grades K-2, MATHgazine Elementary for grades 2-5, MATHgazine Junior for grades 4-8 and the MATHgazine Senior for grades 8-12 which are distributed by the AATM to teachers in Arizona. Emily also works as an Accounting Assistant. Emily will start student teaching in 1st grade this Fall.
Tanner Wolfram is a Fall 2019 graduate (Summa cum Laude) of Barrett, The Honors College at Arizona State University. He holds a major in Physics and minors in both Spanish and Chinese. Since Fall 2019, Tanner has served as a Senior Project Assistant in the Practice, Research, and Innovation in Mathematics Education (PRIME) Group at ASU. He works with the PRIME Group to conduct research on Preservice Teachers Knowledge of Mathematics and Middle School Students’ Algebraic Reasoning Talents, while in addition continuing work that he began as an undergraduate Project Assistant (Spring 2016 – Summer 2019) in the PRIME Center. During his time with the PRIME Group/Center, Tanner has assisted with the NSF-funded Project App Maker Pro (AMP), has edited eight MATHadazzle Puzzle Books, and co-authored four articles in Math Education. He has also authored, a soon-to-be published, Facasumi Puzzle Book. Tanner is a co-editor of the free monthly AATM MATHgazine Primary (grades K-2), MATHgazine Elementary (grades 2-5), Junior (Grades 4-8) and MATHgazine Senior (Grades 8-12). Additionally, Tanner is a Co-Editor of the AATM Journal. In his free time, he plays tennis, table tennis, badminton, and learns more about the stock market. Tanner hopes to be in graduate school at Arizona State University in 2021.
Is it our responsibility to introduce high school students to contemporary mathematics?

Nitsa Movshovitz-Hadar

Abstract
Should high school mathematics teachers give their students a taste of contemporary mathematics? In this article, obstacles and a suggested solution are provided to enable teachers to incorporate contemporary mathematics into their programs.

Introduction
“…One of the big misapprehensions about Mathematics that we perpetuate in our Classrooms is that the teacher always seems to know the answer to any problem that is discussed. This gives students the idea that there is a book somewhere, with all the right answers to all of the interesting questions, and that teachers know those answers. And if one could get hold of the book, one would have everything settled. That's so unlike the true nature of mathematics…” (Henkin, 1981)

What is the true nature of mathematics?
What follows are five illustrative examples.
1. The centuries-old Twin Primes Conjecture (TPC) posits that there are infinitely many consecutive primes, i.e., pairs of primes separated by exactly 2 (e.g., 11,13; 17,19). This conjecture seems to stand contrary to our intuition, rooted in the general tendency of gaps between adjacent primes to become greater as the numbers themselves get larger. On May 13, 2013, a lecturer at the University of New Hampshire showed that you would never stop finding pairs of primes separated by 70 million (or less). He was the first to establish the existence of a finite bound for prime gaps. This finding gave a start to a mathematical race. On Nov 19, 2013, a researcher at the University of Montreal, pushed the gap down to 600 (Meynard, 2013). On April 14, 2014, the gap was reduced to 246 and later to 12, then to 6. How soon will the gap be reduced to 2, thus proving the TPC? – This remains to be seen. (For more details, see Wikipedia Twin Prime.)
2. In 1611 Johannes Kepler stated that even the densest packing of spheres of equal diameter takes no more than ~74% of the space. For more than 400 years, the mathematics community struggled to prove it, in vain. There seemed to be a breakthrough when, on August 10, 2014, a team led by a mathematician at the University of Pittsburgh, announced that their decade-long effort to construct a computer-verified formal proof of the Kepler conjecture, was complete. This 400-year-old best sphere-packing conjecture had finally become a theorem. Furthermore, the long struggle for getting this proof recognized as valid beyond doubts paved the way for a new era of mathematics, in which machines will do the “dirty work” of verifying proofs before their publication in a respectful journal and leave mathematicians free for deeper thinking. (Hales et al., 2017)
3. Take a look at the first 40 elements of the well-known Fibonacci series: \( a_1 = a_2 = 1 \) and for \( n > 2 \) \( a_n = a_{n-1} + a_{n-2} \): 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 11946, 18711, 30657, 49368, 80025, 129393, 209418, 338811, 548229, 887040, 1435269, 2322309, 3757578, 6079887, 9837465, 15917352, 25754817, 41672169, 67426986, 109099155…

Now, let’s examine the occurrence frequencies of the leading digits (the left-most digit) in the part of the sequence shown above. Would you expect a difference?

<table>
<thead>
<tr>
<th>Leading digit</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>11</td>
<td>7</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

Are you surprised? This unique phenomenon of the uneven distribution of the leading figure is known as Benford’s Law. One finds it in many other series such as \( a_n = n! \) and \( a_n = 2^n \). And surprisingly, it is true of much statistical data, as well. Look at the distribution of the leading digit of the altitudes of the top 122,000 populated cities shown in the graph below.

The Benford law was applied to forensic sciences. In 1993, Wayne James Nelson was found guilty of trying to defraud The State of Arizona of nearly $2 million by diverting funds to a bogus vendor (State of Arizona v. Wayne James Nelson CV92-18841.) He selected payments carefully, intending to make them appear random: None of the check-amounts were duplicated, there were no rounded numbers, and all the values included numbers of dollars and cents. The poor guy failed to realize that his seemingly random looking selections, were far from behaving as expected by Benford’s law! (Durtschi et al., 2004.)
4. In 1975, in his well-known “Mathematical Games” column in *Scientific American*, Martin Gardner surveyed all that was known on tessellating the plane with convex polygon tiles. Among others, he referred to regular pentagons that do not tessellate the plane (why?), and to the study of non-regular pentagons that do tessellate. At that time, eight families of pentagons were known to facilitate the tessellation of the plane, and a prominent mathematician claimed that there were no more. However, soon after his paper was published, one of the readers, a computer programmer, discovered the 9th type of pentagons that tessellate the plane, which falsified the mathematician’s claim and reopened the search for more types (assuming there are more). Indeed, two years later, a housewife from California, blending her love for mathematics and her love for art, discovered four additional types of pentagons that tile the plane (Schattschneider, 1978). In 1985, a German student found an additional type, raising the number of types to 14. No others were found for 30 years. Then, in October 2015, the following title appeared in The Guardian: "Attack on the pentagon results in discovery of new mathematical tile." ([https://www.theguardian.com/science/alexs-adventures-in-numberland/2015/aug/10/attack-on-the-pentagon-results-in-discovery-of-new-mathematical-tile](https://www.theguardian.com/science/alexs-adventures-in-numberland/2015/aug/10/attack-on-the-pentagon-results-in-discovery-of-new-mathematical-tile)). The article described the discovery of a 15th type of tiling by a group of young mathematicians from the University of Washington. Yet the general question remained open: Are there additional types of pentagons that can be used to tile the plane? How many? In the summer of 2017, a 37-year-old French mathematician, published a computer proof that there are no more than 15 types of convex pentagonal prototiles, thus bringing the drama to closure (Rao, 2017). (See all 15 types here: [http://www.wolframalpha.com/input/?i=pentagon+tiling](http://www.wolframalpha.com/input/?i=pentagon+tiling)).

5. Perhaps the best illustration of the creative nature of mathematics is numerical. According to MathSciNet, AMS database of reviews, abstracts, and bibliographic information for much of the mathematical sciences literature, “over 125,000 new items are added each year, … [and] over 90,000 reviews are added to the database each year… Reference lists are collected and matched internally from approximately 650 journals.”

**Back to Henkin’s quote that opened this article**

In most cases, students graduate high school without any exposure to contemporary mathematics in the making. They go out to their next stage in life, having the (wrong) impression of mathematics as a “dead-end” boring discipline in which all answers are known (to the teacher), and nothing is left open for them to stimulate their intellectual curiosity or creativity. This impression is largely due to the very nature of mathematics as an accumulative and hierarchical discipline, so that one cannot be expected to understand accurately and deeply the details of contemporary mathematics without learning first the past achievements which never become obsolete. Hence, mathematics curricula in many countries do not reach beyond the 18th-century mathematics, and contemporary mathematics belongs to tertiary education. Covering the mandatory curriculum, preparing students for school graduation exams or university admission requirements, is a stressful job that leaves very little freedom for the teachers to toy with the idea of exposing their students to more than required.
Why should we care?

Is it worth our time to invest in attempting to cope with this seemingly impossible task? After all, there are so many other issues for the mathematics education community to worry about. Well, there are several good reasons to care about the lack of exposure to modern mathematics in school. As a result of it, only a small percentage of high school graduates consider tertiary education in mathematics or in mathematics-related professions, although the job market in these areas is very promising. This, in turn, has an impact on the economy. Moreover, it perpetuates the ignorance of the adult society about mathematics. Rather than being literate about mathematics and its impact on the quality of our lives in the 21st century, most adults could not care less about mathematics once they leave high school. At best, they admire blindly those “weird” ones who chose to become mathematicians, having no idea what it means to do mathematics.

The gap between school mathematics and contemporary mathematics, is it a necessary evil?

Can the high school mathematics teacher be expected to be able to give high school students a taste of contemporary mathematics without harming the progress in teaching the heavily loaded mandatory curriculum? And facing the increased effort it takes to keep up-to-date in contemporary mathematics?

How might it be possible to communicate the true nature of contemporary mathematics to ALL high-school students without harming their progress in learning the heavily-loaded mandatory curriculum? And facing students’ insufficient backgrounds required to delve into the depth of modern mathematics?
A few questions to ponder before we attempt at a solution

How do we make young children curious about written words? -- By reading a story to them and pointing at the printed words; By teaching them how to read and write; By exposing them to classic books. We do not do it only because reading and writing are essential for functioning in the daily life of the adult world. Nor do we do it because we expect all of them to become poets or book authors. We do want to open for them the door to reading the literature, because it is a great part of our culture, deeply rooted in generations of human creativity.

How do we make children like music? -- By turning on some machine that plays some music, we believe would be pleasant to their ears; By teaching them how to sing a song; By suggesting that they learn to play an instrument. We do not do it because they need it for any practical purpose. We do not expect all of them to become professional musicians. We do want to open a window to music for them because it is a rich part of our culture.

How do we make students appreciate the performing arts? -- By taking them to the theater to watch a play, we believe it would touch their soul; By encouraging them to put on a costume and perform some role and enjoy it. We do not do it because we aim at training them as actors or dancers. We do it because we want to expose them to this wonderful part of human culture, of human creation over generations.

How can we make our students curious about mathematics, like mathematics, enjoy it, and appreciate its impact without expecting many of them to become mathematicians? Have we gone a bit too far with the emphasis on learning-by-doing and problem-solving? Sometimes a piece of mathematics you read or hear about can be so neat and elegant that you feel the AHA effect, and it makes you want to shout, "Got it!" (even if you did not discover it yourself!) Do you enjoy hearing a clever solution to an interesting mathematics problem you were not able to figure out on your own? Do you find a well-written expository paper interesting and fulfilling? Do you like a vivid presentation of a counterintuitive mathematical result that is not familiar? Why deprive school students of these pleasures?
The Math News Snapshots Project

The Math News Snapshots (MNS) Project is built on the premise that high school students should not be deprived of these pleasures. They should have the opportunity to learn about contemporary research in mathematics as part of their education, because it is a part of human culture. Quite often, the hierarchical nature of mathematics does not make contemporary mathematics accessible to students’ active learning. On the other hand, ignoring the vivid and prolific nature of contemporary mathematics leaves students thinking that all mathematics has been “done” already. Consequently, students often do not understand the work of living mathematicians. We argue that learning about new findings in mathematics and new applications of mathematics is as necessary for widening students’ mathematical horizons as having them experience the joy of problem-solving, firsthand.

MNS lessons provide a glimpse into contemporary mathematics tailored to be accessible to high school students. Each MNS is in the form of a PowerPoint presentation. It is focused on a single result, along with its background, implications, and applications where relevant. (Please see the project website at [https://mns.org.il/](https://mns.org.il/) for examples.) Interweaving MNSs one at a time throughout the teaching of the high school curriculum is worth trying. It can have a significant effect on the way students perceive the true nature of mathematics.

Note: The MNS project started in Israel in 2012 and has become well established there. In 2017-2019, a longitudinal study supported by the Israeli Ministry of Science was conducted to investigate the long-term effects of introducing high school students to mathematical news on a regular basis throughout their three years of high school (Movshovitz-Hadar et al., 2019). Currently, our MNS team is looking for pilot sites and partners who are interested in bringing MNSs to U.S. classrooms. Please contact Nitsa Movshovitz-Hadar at nitsa@technion.ac.il to learn more about the project.

Resources and recommendations for further reading

Math New Snapshots Website: [https://mns.org.il/](https://mns.org.il/)


References


Nitsa Movshovitz-Hadar, Ph.D., is Professor Emerita at the Technion – Israel Institute of Technology. She began her career as a high school teacher, and after 12 years, decided to devote her time to teacher education, curriculum development, and related research. In 1975, she received her Ph.D. at the University of California - Berkeley. After completing this program, she returned to the Technion to establish the Research and Development Center for Mathematics Education. She has also served as Director of MadaTech, the Israel National Museum of Science, and is a member of the advisory committee for MOMATH. Nitsa has led several research projects, including the most recent, Math News Snapshots project. To date, this project has developed 24 PowerPoint presentations, each focusing on one new result in mathematics, and making those accessible to high school students via their classroom teachers.
From Euclid to Math Competitions to Math Classes

Richard Kalman

Abstract

Math facts are just the vessel we use to develop students’ problem-solving talents, our primary objective. This article offers some ideas about making math come alive for all students.

Euclid got it wrong. Huh?
Well, okay. He got it right. But we got it wrong for some 2,300 years!
Here’s the story.

Euclid organized everything he knew about geometry into a logical order. Starting from definitions, he examined basic assumptions, classifying some as axioms and the others as postulates. This provided the foundation on which he proved theorem after theorem, thereby showing that every property followed logically from his foundation.

Euclid’s extraordinary development has been followed slavishly for more than two millennia. It is absolutely “brilliant” by providing a complete and logical review of all we know about geometric shapes. The problem is that teachers routinely have had to tell students, “All this will make sense after eight or ten weeks.” Some students began to see the overall picture quickly, while others never did.

I believe this rather dry approach lost the magic inherent in mathematics. How many teachers really looked at why people like Archimedes, Newton, Pascal, and even their better students, spent so much time and energy studying mathematical relationships?

Most mathematics, perhaps even all, developed as a result of challenges. In many cases, one person asked another to solve a problem. In some cases, someone challenged himself or herself. Either way, it wasn’t the knowledge that triggered the desire to explore. It was the excitement of meeting and beating the challenge itself.

Problem solving is really what we teach. Math principles and facts are merely the vehicle. Here are suggestions that worked for me.

1. To begin a lesson, I assigned a problem in which every step but one reviewed previous work. That one step depended upon the concept yet to be taught that day, presented in an intuitive way. Any student who solved the problem would show and explain the solution to everyone, and answer all questions posed by fellow students. If the class didn’t challenge the student to justify the new concept, I would.

2. Once this was done, I’d assign more problems, each of which was designed to use the concept as differently as possible, while also reviewing a wide variety of previous content, both old and recent. As before, I’d get out of their way and have students present their solutions and field questions from classmates. Where I could, I encouraged students to present many different solutions to a given problem.

3. As the period ended, I often offered an appropriate challenging problem to be solved for extra credit.
Math team competitions provided an excellent source for these problems. Students who accepted the challenge handed in their solutions at the next class meeting. Two points were awarded for attempting a solution, two points for a correct answer, and up to six points for the quality of the solution itself.

Here are two such extra credit problems, each designed to stretch student thinking:

1. Two circles are concentric. Two distinct chords of the larger circle are each tangent to the smaller circle. If \( a^2 - 2a - 12 \) represents the length of one chord, and \( 24 - 7a \) represents the length of the other chord, what is the value of \( a \)?

2. Identify all integer values of \( n \) for which \( \frac{7n - 26}{n - 5} \) is an integer. What is the product of the greatest and least values of \( n \)?

**Solutions**

1. **Answer:** \(-9\) This problem is designed to convince students to check their work. Many will automatically reject the negative value of \( a \) and accept the positive value. As it “turns out,” only the negative value produces positive lengths for the chords. Students are very likely to remember a “gotcha” moment like this. It may convince them to check solutions against the original problem.

   **Solution:** Since both chords are the same distance from the center, the chords are congruent. Thus, \( a^2 - 2a - 12 = 24 - 7a \), and \( a = 4 \) or \(-9\). However, a chord must have a positive length. Replacing \( a \) by 4 produces the impossible length for each chord of \(-4\). Replacing \( a \) by \(-9\) produces a length for each chord of 87. Therefore, only \( a = -9 \) satisfies the problem.

   *Note: Without loss of generality, make the chords parallel. Seeing relationships becomes easier. In fact, thinking of the chords as rotating about the center allows us to regard them as two positions of the same chord.*

2. **Answer:** \(-56\) This problem highlights four goals: 1) Students must apply the arithmetic concept to algebra of changing an improper fraction into a mixed number. 2) They must understand what numbers the algebraic statement \( n - 5 \) represent. 3) They must realize that a positive number also has negative factors, and 4) like in problem 1, they must continue an additional step. As Yogi said, “It ain’t over until it’s over.”

   **Solution:** \((7n - 26) \div (n - 5)\) can be separated into the sum of two fractions: \((7n - 35)/(n - 5) + (+9)/(n - 5)\). Simplify this algebraic fraction into the algebraic expression \(7 + (9)/(n - 5)\). The question now becomes: “Which six values of \( n \) in \((n - 5)\) produce a divisor of 9?” The six divisors are 9, 3, 1, \(-1\), \(-3\) and \(-9\). Therefore, the six values of \( n \) are 14, 8, 6, 4, 2 and \(-4\). The product of the least (-4) and the greatest (14) is \(-56\).
Richard Kalman was very involved in mathematical competitions from 1953 to 2014 as a student, teacher, math team coach, problem author, lecturer, book editor, and executive director, for all levels from grade four through twelve. Nationally, he was best known for his leadership positions with the American Regions Math League and the Math Olympiads for Elementary and Middle Schools. He now lives in Florida where he lectures locally on history and musicals.
Abstract
For hundreds of years mathematicians have studied unusual types of numbers. Those specializing in the study of strange numbers are *number theorists*. You may be familiar with some of these strange numbers, but others, well you may not have heard of them before. Ready?
**EMIRP Numbers:**

An **EMIRP** number is a PRIME number with two or more digits that, when the digits are reversed, is also a prime number. (Note: EMIRP is PRIME spelled backwards!)

Example: 17 is a prime number. Reverse it digits and 71 is also a prime number. So, 17 is an EMIRP Number (and its partner is 71).

Which of these are EMIRP numbers? If so, record its partner. For the ones that are not EMIRP, show their factors.

1) 13  
2) 19  
3) 37  
4) 79  
5) 107  
6) 171  
7) 199  
8) 543

**EMIRP Numbers Solutions:**

1) EMIRP partner is 31
2) Not EMIRP: factors of 91 are 7 and 13
3) EMIRP partner is 73
4) EMIRP partner is 97
5) EMIRP partner is 701
6) Not EMIRP: factors of 171 are 3, 9, 19 and 57
7) EMIRP partner is 991
8) Not EMIRP factors of 543 are 3 and 181
Circular Primes:

A **Circular Prime** number is a multi-digit prime number that, after moving its digits around in a “circular” fashion (last digit becomes the first digit, and so on), numbers that result are also prime numbers.

Example: 113 is a circular prime number because:
- When you move its last digit 3 to the first digit, you get 311, also a prime number.
- Move its last digit 1 to the first digit, you get 131, also a prime number.
- Move the last digit of 1 to the first digit, you get 113, the original prime number.

Which of these prime numbers are circular primes? Describe how you made your decision.

1) 197  2) 213  3) 241  4) 357  5) 377  6) 1193

7. The number cannot be a circular prime if it contains certain digits. What are those digits?

Circular Primes Solutions:

1) Yes. 197 is a prime number. It circles to 719 to 971 to 197. All numbers are prime numbers.

2) No. 213 is not a prime number. Factors are itself, 1, 3 and 71.

3) No. 241 is prime number. It circles to 124 which is not a prime number. Factors of 124 are 1, itself, 2, 4, 31, and 62.

4) No. 357 is a prime number. It circles to 735 which is not a prime number. Factors of 735 are 1, itself, 3, 5, 7, 15, 21, 35, 49, 105, 147, and 245.

5) No. Factors of 377 are 1, itself, 13 and 29.

6) Yes. 1193 is a prime number. It circles to 3119 to 9311 to 1931 to 1193. All numbers are prime numbers.

7) 0, 2, 4, 5, 6, 8
Mystery Numbers:

A Mystery Number is a number that is the sum of two numbers and those two numbers are the reverse of one another.

Example: 121 is a mystery number because it is the sum of 47 and 74.
It is also the sum of 29 and 92.

Determine if each of these is a mystery number. Show the two reversed numbers that add to the number. (Note: There are more solutions.)

1) 22  2) 77  3) 101  4) 110  5) 132  6) 444

Mystery Numbers Solutions:
1) Yes  22 = 11 + 11
2) Yes  77 = 34 + 43; 25 + 52; 16 and 61
3) No
4) Yes  110 = 37 + 73; 28 + 82; 19 + 91
5) Yes  132 = 48 + 84; 39 + 93
6) Yes  444 = 123 + 321
**Powerful Numbers:**

A Powerful Number is a positive number and the product of a square number and a cubic number.

Example: 4 is Powerful. It is the product of $2^2$ and $1^3$ ($4 \times 1 = 4$)

1) What are the Powerful Numbers 1 through 100, as well as 4? Show their products.
2) What is the least four-digit Powerful Number?
3) Show that 59,049 is a Powerful Number.

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**Powerful Numbers Solutions:**

1) Powerful Numbers 1 through 100.

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2. Least Four Digit Powerful Number $1000 = 1^2 \times 10^3$

3. 59,049 = $9^2 \times 9^3$
Happy Numbers:

Sophie Germain, a famous French mathematician, physicist, and philosopher, discovered that all numbers can be categorized as “happy” or “unhappy.”

How to figure out if a number is Happy: Pick a number with two or more digits, then:
- Square each of its digits and add.
- Square each digit in that sum and add.
- Repeat until the sum is a single-digit number.
- If the final sum is 1, then the original number is Happy!

Example: 44 is a Happy Number because:

\[ 4^2 + 4^2 = 16 + 16 = 32 \]
\[ 3^2 + 2^2 = 9 + 4 = 13 \]
\[ 1^2 + 3^2 = 10 \]
\[ 1^2 + 0^2 = 1 \]

Show why these numbers are Happy Numbers.
1) 19  2) 320  3) 802

Happy Numbers Solutions:
1) 19 is \( 1^2 + 9^2 = 82; 8^2 + 2^2 = 68; 6^2 + 8^2 = 100; 1^2 + 0^2 + 0^2 = 1 \)
2) 320 is \( 3^2 + 2^2 + 0^2 = 13; 1^2 + 3^2 = 10; 1^2 + 0^2 = 1 \)
3) 802 is \( 8^2 + 0^2 + 2^2 = 68; 6^2 + 8^2 = 100; 1^2 + 0^2 + 0^2 = 1 \)
Perfect Numbers:

A number is **Perfect** if it is equal to the sum of its proper divisors. Proper divisors do not include the number itself.

Example: 6 is a **Perfect** Number. Its proper divisors are 1, 2, and 3. The sum of those divisors is 6.

Show that these numbers are Perfect Numbers. Identify their factors and sums.

1) 28  2) 496  3) 8,128

Perfect Numbers Solutions:

1) 1, 2, 4, 7, and 14. Sum is 28.
2) 1, 2, 4, 8, 16, 31, 62, 124, and 248. Sum is 496.
3) 1, 2, 4, 8, 16, 32, 64, 127, 254, 508, 1016, 2032, and 4064. Sum is 8,128.
Abundant Numbers:

A number is **Abundant** when the sum of its proper divisors is greater than the number itself. Example: 12 is an abundant number because its proper divisors are 1, 2, 3, 4, and 6. The sum of those proper divisors is 16 and 16 is greater than 12.

1) The greatest two-digit abundant number is 96. What are its proper divisors? What is the sum of those divisors?

2) The least three-digit abundant number is 100. What are its proper divisors? What is the sum of those divisors?

3) The smallest odd abundant number is 945. What are its proper divisors? What is the sum of those divisors?

Abundant Numbers Solutions:

1) 96 proper divisors: 1, 2, 3, 4, 6, 8, 12, 16, 24, 32 and 48. Sum is 156.
2) 100 proper divisors: 1, 2, 4, 5, 10, 20, 25, and 50. Sum is 117.
3) 945 proper divisors: 1, 3, 5, 7, 9, 15, 21, 27, 35, 45, 63, 105, 135, 189, and 315. Sum is 975.
Deficient Numbers:

A number is Deficient if the sum of its proper divisors is less than the number itself. Note: All prime numbers are deficient. However, there are non-prime numbers that are deficient. Example: 14 is a Deficient number because the its proper divisors are 1, 2, and 7. Their sum is 10 which is less than the number itself.

For each of these deficient numbers, identify the proper divisors and sum.

1) 21  2) 46  3) 105

Deficient Numbers Solutions
1) 21 proper divisors: 1, 3, and 7. Sum is 11.
2) 46 proper divisors: 1, 2, and 23. Sum is 26.
3) 105 proper divisors: 1, 3, 5, 7, 15, 21, and 35. Sum is 87.
Carol Numbers:

Carol Numbers were first studied by Cletus Emmanuel, a professor of mathematics at the University of Virgin Islands. He named the numbers after his friend, Carol G. Kirnon. A Carol number is an integer of the form $4^n - 2^{(n+1)} - 1$. Replace $n$ with positive numbers to identify Carol Numbers.

Example: Replace $n$ with 1, and compute $4^n - 2^{(n+1)} - 1$.

\[ n = 1 \quad \text{The first Carol Number is} \; 1 \]

1) Replace $n$ with 2 to compute the second Carol Number.

2) Replace $n$ with 3 to compute the third Carol Number.

3) Replace $n$ with 4 to compute the fourth Carol Number.

Carol Numbers Solutions:

1) 7  2) 47  3) 223
Unusual Numbers:

A number is **Unusual** if it is a natural number \( (n) \) whose greatest prime factor is greater than \( \sqrt{n} \). Note that all prime numbers are **Unusual** since their greatest prime factor is the number itself. There are an infinite number of Unusual numbers that are not prime.

Example: 15 is unusual. Its greatest prime factor is 5 and \( \sqrt{15} = 3.87 \). 5 is greater than 3.87.

Show why each of these numbers is **Unusual**.

9) 44  
2) 57  
3) 76  
4) 99

Which of these numbers is **Unusual**? Give a rationale for each.

5) 80  
6) 83  
7) 85  
8) 90  
9) 96  
10) 97

**Unusual Number Solutions:**

1) Greatest prime factor of 44 is 11. Square root of 44 is 6.63. 11 is greater than 6.63.
2) Greatest prime factor of 57 is 19. Square root of 57 is 7.55. 19 is greater than 7.55.
3) Greatest prime factor of 76 is 19. Square root of 76 is 8.72. 19 is greater than 8.72.
4) Greatest prime factor of 99 is 11. Square root of 99 is 9.95. 11 is greater than 9.95.
5) Greatest prime factor of 80 is 5. Square root of 80 is 8.94. 5 is less than 8.94 so 80 is not unusual.
6) 83 is a prime number. **83 is an unusual number.**
7) Greatest prime factor of 85 is 17. Square root of 85 is 9.22. 17 is greater than 9.22, so, **85 is an unusual number.**
8) Greatest prime factor of 90 is 5. Square root of 90 is 9.49. 5 is less than 9.49, so 90 is not an unusual number.
9) Greatest prime factor of 96 is 3. Square root of 96 is 9.80. 3 is less than 9.80, so 96 is not an unusual number.
10) 97 is a prime number. **97 is an unusual number.**
**Weird Numbers:**

A number is **Weird** if it is abundant (sum of its proper divisors – not including the number itself – is greater than the number) and no subset of its proper divisors add to the number itself.

Example: 70 is a weird number. In fact, it is the least weird number! Its proper divisors are 1, 2, 5, 7, 10, 14 and 35. The sum of those proper divisors is 74, and 74 is greater than 70. However, no subset of those proper divisors has a sum of 70. So, 70 is weird! (Note: The next greatest weird number is 836.)

Show why these numbers are not weird.

1) 12  2) 24  3) 96  4) 100  5) 204

**Weird Numbers Solutions:**

1) Proper divisors of 12 are: 1, 2, 3, 4, and 6. The sum of those proper divisors is 16 and 16 is greater than 12. However, the sum $2 + 4 + 6 = 12$. So, 12 is not weird.

2) Proper divisors of 24 are: 1, 2, 3, 4, 6, 8, and 12. The sum of those proper divisors is 36 and 36 is greater than 24. However, the sum $4 + 8 + 12 = 24$. So, 24 is not weird.

3) Proper divisors of 96 are 1, 2, 3, 4, 6, 8, 12, 16, 24, 32 and 48. The sum of those proper divisors is 156 and 156 is greater than 96. However, the sum $4 + 8 + 12 + 16 + 24 + 32 = 96$. So, 96 is not weird.

4) Proper divisors of 100 are: 1, 2, 4, 5, 10, 20, 25, and 50. The sum of those proper divisors is 117 and 117 is greater than 100. However, the sum $5 + 20 + 25 + 50 = 100$. So, 100 is not weird.

5) Proper divisors of 204 are: 1, 2, 3, 4, 6, 12, 17, 34, 51, 68, and 102. The sum of those proper divisors is 300. However, the sum $34 + 68 + 102 = 204$. So, 204 is not weird.
Harmonic Numbers:

Harmonic Numbers are the sums of the reciprocals (1/n) of the natural numbers, beginning with 1. The name Harmonic comes from the concept of overtones or harmonics in music. Harmonic Numbers were studied in 1668 by the German mathematician Nicolaus Mercator, and in 1689 by the Swiss mathematician and physicist, Jacob Bernoulli.

The first Harmonic Number is: \(1\).
The second Harmonic Number is: \(1 + \frac{1}{2} = 1.5\)
The third Harmonic Number is: \(1 + \frac{1}{2} + \frac{1}{3} = 1\frac{5}{6} \approx 1.83\) to the nearest hundredth.
The fourth Harmonic number is: \(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 1 + 1\frac{3}{12}, \text{ or } 2\frac{1}{12} \approx 2.08\) to the nearest hundredth.

Show each of these Harmonic numbers in both a list of fractions to add and their decimal sums. Round decimals to hundredths.

1) Fifth Harmonic number
2) Sixth Harmonic number
3) Tenth Harmonic number

Harmonic Numbers Solutions:

1) Fifth Harmonic number: \(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\) This is equal to the fourth Harmonic number 2.08 plus \(1/5\), or 0.2; \(2.08 + 0.2 = 2.28\). **The fifth Harmonic number is 2.28**
2) Sixth Harmonic number: \(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\). This is equal to the fifth Harmonic number 2.28 plus \(1/6\), or \(2.28 + 0.17 = 2.45\). **The sixth Harmonic number is 2.45**
3) Tenth Harmonic number: \(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\). This is equal to the sixth Harmonic number 2.45 + 0.14 + 0.13 + 0.11 + 0.1 = 2.93. **The tenth Harmonic number is 2.93**.
Proth Numbers:

A Proth Number is a positive integer of the form \( p = k \cdot 2^n + 1 \) in which \( k \) is an odd positive integer, \( n \) is a positive integer, and \( 2^n > k \).

Example: 3 is a Proth number because

\[
p = k \cdot 2^n + 1 \\
3 = 1 \times 2^1 + 1
\]

Show why these numbers are Proth numbers.

1) 9  
2) 17  
3) 25  
4) 33  
5) 41  
6) 49

**Proth Numbers Solutions:** (Solutions may vary.)

1) \( 1 \times 2^3 + 1 = 9 \)  
2) \( 1 \times 2^4 + 1 = 17 \)  
3) \( 3 \times 2^3 + 1 = 25 \)  
4) \( 1 \times 2^5 + 1 = 33 \)  
5) \( 5 \times 2^3 + 1 = 41 \)  
6) \( 3 \times 2^4 + 1 = 49 \)
Carole Greenes, Ed.D. is Professor Emerita, Mathematics Education in the Mary Lou Fulton Teachers College at Arizona State University. Prior to that, she was Director of the Practice Research and Innovation in Mathematics Education (PRIME) Center, and Professor of Mathematics Education in the Ira A. Fulton Schools of Engineering and the College of Liberal Arts and Sciences at Arizona State University. Currently, she directs the PRIME Group that develops books of challenge problems for students, grades K – 12, and conducts research projects (Preservice Teachers Knowledge of Mathematics and Middle Students’ Algebraic Reasoning Talents.) Carole is author of more than 350 books for PreK-16 and teachers; 79 articles; six mathematical musicals; and two histories of mathematics in story and song. She is editor of the Arizona Association of Teachers of Mathematics journal, OnCore, and author of the online monthly free MATHgazine Senior (grades 8-12), MATHgazine Junior (grades 4-8), MATHgazine Elementary (grades 2-5) and MATHgazine Primary (grades K-2). In 2003, Greenes was inducted into the Massachusetts Mathematics Educators’ Hall of Fame. In 2011, she received the NCSM Ross Taylor/Glenn Gilbert National Leadership Award in Mathematics Education. In 2016, she received the Copper Apple Award for Leadership in Mathematics in Arizona, and in 2018 she received the National Council of Teachers of Mathematics Lifetime Achievement Award.
Tanner Wolfram is a Fall 2019 graduate (Summa cum Laude) of Barrett, The Honors College at Arizona State University. He holds a major in Physics and minors in both Spanish and Chinese. Since Fall 2019, Tanner has served as a Senior Project Assistant in the Practice, Research, and Innovation in Mathematics Education (PRIME) Group at ASU. He works with the PRIME Group to conduct research on Preservice Teachers Knowledge of Mathematics and Middle School Students’ Algebraic Reasoning Talents, while in addition continuing work that he began as an undergraduate Project Assistant (Spring 2016 – Summer 2019) in the PRIME Center. During his time with the PRIME Group/Center, Tanner has assisted with the NSF-funded Project App Maker Pro (AMP), has edited eight MATHadazzle Puzzle Books, and co-authored four articles in Math Education. He has also authored, a soon-to-be published, Facasumi Puzzle Book. Tanner is a co-editor of the free monthly AATM MATHgazine Primary (grades K-2), MATHgazine Elementary (grades 2-5), Junior (Grades 4-8) and MATHgazine Senior (Grades 8-12). Additionally, Tanner is a Co-Editor of the AATM Journal. In his free time, he plays tennis, table tennis, badminton, and learns more about the stock market. Tanner hopes to be in graduate school at Arizona State University in 2021.

Emily Branam is a senior in the Mary Lou Fulton Teachers College at Arizona State University, studying Elementary Education with a STEM focus. She is a Project Assistant in the PRIME Group at ASU, a co-author of four articles, a co-editor of the AATM OnCore Journal, and a co-author of the MATHgazine Primary for grades K-2, MATHgazine Elementary for grades 2 – 5, MATHgazine Junior for grades 4-8 and the MATHgazine Senior for grades 8-12 which are distributed by the AATM to teachers in Arizona. Emily also works as an Accounting Assistant. Emily will start student teaching in 1st grade this Fall.