Dear AATM Colleagues,

Welcome to the spring 2019 issue of OnCore. We are pleased to have 7 articles. The articles are ordered by grade levels.

*Visualizing Factors and Multiples* activities are designed to enhance upper elementary school students’ understanding of factors, multiples, least common multiple, and greatest common factor. Additional activities are provided in hand-out type format for additional practice. *Inquiry-Based Mathematics Teaching: Does It Really Take Too Much Time?* focuses on ways to use structured inquiry to connect an inquiry approach to direct instruction in order to enhance the learning of mathematics. Three activities are used to illustrate this connection. *Is The Order of Operations Prescribed by Some “Higher Power?* addresses the use of parentheses to indicate order of operations versus the prescribed order of operations (parentheses, exponentiation, multiplication, division, addition, subtraction).

The next four articles focus on uses of technology in the teaching of mathematics. *Constructing Triangles with Technology: Modeling Important Concepts in Geometry* highlights the value of using Google Maps and GeoGebra to create and analyze triangles. Three activities, involving angle measure and similarity, trigonometry and area, and Laws of Sines and Cosines, are described. *Commentary: Computational Thinking-It’s Time* stresses the need for all to learn to communicate with their digital devices. The article presents an easy to understand introduction to coding. *Evaluate Apps for Instruction, Practice and Assessment* focuses on how apps can be analyzed for educational purposes, and presents the evaluations of a set of apps, selected by and reviewed by high school teachers and students. Reviews include how those apps may be incorporated into instruction and serve as valuable aids in enhancing problem solving and communication. In the final article, *Animations: Windows to A Dynamic Mathematics*, the author highlights the advantages of using animations to improve student conceptual understanding and promote communication/conversation skills. Animations can be accessed online by the reader.

We hope that you find these articles helpful.
Please consider contributing an article to our fall 2019 OnCore magazine.

Very best wishes for a happy spring,

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Visualizing Factors and Multiples
Grace A. Pestridge, Megan Mullenmeister, and Terri L. Kurz

Abstract
According to the Common Core Standards for Mathematics (NGA & CCSSO, 2010), students should gain familiarity with factors and multiples, and the concepts of least of common multiple and greatest common factor, beginning in the fourth grade. In this article, successful approaches to teaching this content are described. Handouts are provided to help teachers implement the activities with their students.

The concepts of factors and multiples are central to the study of mathematics in elementary and middle school mathematics. By the completion of Grade 4, students should be able to figure out factors and multiples of numbers, and identify prime and composite numbers (NGA & CCSSO, 2010). Concepts of Greatest Common Factor (GCF) and Least Common Multiple (LCM) are important because they are useful in the adding and subtracting of fractions, factoring in Algebra, and graphing in Trigonometry (Zazkis & Truman, 2015).

Color Tiles
Color tiles can be used to support the development and understanding of prime and composite numbers, as well as factors and multiples of numbers. Color tiles are flat square inch tiles, usually available in red, yellow, blue and green. Using these tiles, rectangles can be created to help students see the difference between prime and composite numbers. For example, consider the numbers 9, 10 and 11, and the different rectangles that indicate features of each number (Figure 1). Any number that yields only two rectangles (like 11) is a prime number. Any number that yields more than two rectangles (like 9 and 10) is a composite number. The number of factors is determined by the number of different rectangles. For example, 10 has four rectangles and four factors (1, 2, 5, and 10). The number 9 has a unique feature because it is a square number. Square numbers can be determined if one of the rectangles is a square.
Figure 1. All Rectangles (using color tiles) for the Numbers 9, 10 and 11
An extension to the color tile activity can include exploration of the concept of multiples. If a rectangle for a number is doubled, tripled, or quadrupled, for example, then the multiples are determined for that number. For example, consider the number 9 and its $9 \times 1$ rectangle (length by width). The width of that rectangle can be doubled creating a rectangle measuring $9 \times 2 = 18$ (18 is a multiple of 9); $9 \times 3 = 27$ (27 is a multiple of 9); and so on (Figure 2).

$9 \times 1 = 9$

$9 \times 2 = 18$

$9 \times 3 = 27$

Figure 2. Multiples of 9 by Adding Rows to the Original Rectangle

After students understand how the dimensions and number of rectangles connect to the number of factors and multiples, several questions that prompt thinking and communication can be posed for group exploration, as for example:

- Is 1 a prime or composite number? How do you know? Use the rectangles in your response.
- If a number has a rectangle that is a square, is that number always composite? How do you know?
- Using color tiles, find all prime numbers less than or equal to 50.
- What number less than or equal to 100 has the greatest number of factors? Is it a square number?
- Do greater numbers always have greater numbers of factors? How did you decide?
- What is the difference between a factor and multiple?

Once students develop understanding of factors and multiples, they can begin to explore Greatest Common Factor and Least Common Multiple. In the next two activities, visual models for finding GCFs and LCMs are presented.
Bar Models

Color rods (Cuisenaire Rods) or bars are often used to support instruction of fractions. They also may be used to help students visualize factors and multiples, including the GCF and LCM. A template for the bar models can be found in Appendix A. The template may be printed on cardstock. The bars are 2 centimeters in width, and lengths vary based on colors (Figure 3).

Bar models are particularly useful for finding the GCFs and LCMs of numbers less than or equal to 24. First, guide students through the location of factors of a number using the bar models. Begin with 12 (gray bar). The factors can be found by determining how many bars of the same color, can be lined up to be the same length as the gray bar (Figure 4). The factors 1 (red), 2 (orange), 3 (yellow), 4 (light green), 6 (magenta) and 12 (gray) are easily identifiable. The factors of 9 are shown in Figure 5.
To identify factors of numbers greater than 12, you will need to instruct students to draw a line of that length, for example 15 cm, and identify single-color bars that can fit that length: 1 (red), 3 (yellow), 5 (dark green) and 15 (no color)).

Finding the GCF of Two Numbers:

For example, to find the GCF of 12 and 9, the longest bar to be used repeatedly to fit across both lengths, is the yellow bar (3). See Figure 4 for the factors of 12 and Figure 5 for factors of 9. Although both red and yellow bars can be used to make the two lengths, the yellow bar is the longest so, the GCF is 3. (Note: 1 is also a shared factor, but not the GCF).

The bars may also be used to find the LCM. For example, to figure out the LCM of 3 and 7, have students build a train of bars of each color until the two trains are the same length. The total length of each bar is the LCM = 21 (Figure 6).

In Appendix B, there is a handout that can be used to explore these concepts with students.
In order for students to fully understand factors, multiples, GCFs and LCMs, it is critical that they be provided with a variety of lessons that afford them with opportunities to explore the concepts more deeply. The activities described in this article provide students with that opportunity.

References


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Appendix A

Template for bar models (to scale)
Appendix B

Bar Model Worksheet

1. Use Bar Models to find the GCF of 12 and 8. Shade your solution below:

   GCF =

2. Find the LCM of 3 and 4 using bar models.

   LCM =

   LCM =

   LCM =
3. Find the GCF and LCM of 9 and 6 using bar models.

\[
\begin{array}{c|c}
\text{Bar Model for 9} & \text{Bar Model for 6} \\
\hline
\end{array}
\]

GCF=________    LCM=_________

4. There are 9 boys and 12 girls in Ms. West’s 3rd grade class. She wants to place the students into groups of only boys and only girls. Each group must have the same number of students (with no extras). What is the greatest number of groups Ms. West can make? Use Bar Models to show your work.

5. Patrick is building a wall that is two rows tall. He uses cinder blocks that are 3 feet long for the bottom row of the wall. He uses cinder blocks that are 7 feet long on the top row of the wall. Patrick does not cut any of his cinder blocks. The bottom row and the top row are the same length. If Patrick’s wall is < 40 feet long, how long could it be? Use Bar Models to show your work.

6. Sammy takes piano lessons every 6 days. He takes swimming lessons every 8 days. If Sammy had both a piano lesson and a swim lesson today, how many more days will it be before he has both lessons on the same day again? Use Bar Models to show your work.
Inquiry-Based Mathematics Teaching: Does It Really Take Too Much Time?

Barbara Kinach

Abstract

Despite educators’ acknowledgement of the need to move toward inquiry-based teaching methods that yield a more robust understanding of mathematics, resistance to inquiry is prevalent. A major complaint is that inquiry takes too much time. This article proposes the use of structured inquiry to bridge the divide between an inquiry approach that yields the desired concept-based understanding of mathematics and the direct-instruction approach. Three examples of structured-inquiry activities, which are highly scaffolded and student-thinking centered, are presented.

In the elementary and secondary pre-service mathematics methods courses that I teach, a major challenge is undergraduates’ weak understanding of mathematics. Typically, these future teachers learned mathematics by memorizing rules without understanding why they work.

In my courses, students view the National Council of Teachers of Mathematics (NCTM) video series: Teaching and Learning Mathematics with the Common Core (https://www.nctm.org/Standards-and-Positions/Common-Core-State-Standards/Teaching-and-Learning-Mathematics-with-the-Common-Core/). After viewing the videos, students described their backgrounds in mathematics.

Pre-service Teacher 1: “While I cannot speak for everyone, I can confidently say that when growing up, there were several students who did not consider themselves to be good at math. I would include myself in this group. Mathematics was taught so procedurally that once given a math problem outside of the procedure, no one could solve it. This could be a very simple problem but the conceptual understanding of mathematics had not been built, and therefore, solving every real world problems was difficult. In the video, Mathematics in the Early Grades, the video explores how Common Core Standards have pushed students to understand the reasoning behind the mathematical problem that they are working on.”

Pre-service Teacher 2: “After watching the Mathematics in the Early Grades and the Developing Mathematical Skills in Upper Elementary Grades videos, I felt encouraged. For me math was always intimidating, but while watching these videos I felt determined to teach math in a way that will inspire students. I learned that my students would have an easier time with math if I were encouraging them to solve math problems using multiple ways. Helping
students be visual, and think of a problem in a broader entertaining way, will help leave a positive experience.”

**Pre-service Teacher 3:** “The videos touched on how in past years, mathematics was taught procedurally rather than conceptually. This was the case for my own learning of mathematics throughout elementary school and middle school. Being able to solve mathematical equations procedurally is important, but if students at all ages are just following steps to solve math problems, they are not truly learning how to DO math. Conceptual understanding of math has become the initiative for teachers across all grade levels, especially in the early elementary grades. Teaching young students how to think conceptually by going from concrete examples, to pictorial examples, to the abstract math representation, is creating more powerful thinkers at younger ages, which is setting thousands of students up for success as they grow older.”

Hands-on manipulative activities, combined with readings, video viewings, and interactions with K-12 students, generate a positive mindset and willingness and desire among my pre-service teachers to teach mathematics conceptually using inductive inquiry teaching methods that are far removed from the ways they learned mathematics. A problem arises when they hear statements, such as: 1) *Inquiry takes too much time;* 2) *It doesn’t work;* or 3) *Last year, a teacher who used inquiry teaching methods got fired because 50% of her students did not pass the AZ Merit test.*

The activities that follow are examples of inquiry-based instruction that introduces mathematical rules, formulas, relationships, and concepts in meaningful ways. Structured-inquiry activities are highly scaffolded, student-thinking centered, and essentially, inductive learning opportunities. These activities capitalize on the generative power of sequenced patterns to foster insight into abstract mathematical ideas through generalization. Structured-inquiry activities rely on students’ abilities to notice, extend, and generalize the patterns within a sequence of similarly structured representations. These activities, that can be conducted within a 50-minute class period, promote deep conceptual understanding of the mathematical notions and processes learned (Boaler, 2018).

**Inquiry-based mathematics teaching**

Inquiry-based mathematics is a style of thinking in which the investigator-learner questions the problem context looking for patterns that can be generalized to form a meaningful and deep concept-based (Wathall, 2018) understanding of a mathematical rule, formula, relationship, or concept. The *Common Core Mathematics Standards* (NGA & CCSSO, 2010) advocates inquiry-style mathematics teaching practices specifically when students are learning new mathematical ideas. By contrast, direct-instruction teaching methods call upon students who are learning new concepts to first duplicate the procedures modeled by the teacher, and then apply them.
Structured-Inquiry Activity for Generalizing a Mathematical Rule

To illustrate further, consider a direct-instruction approach to teaching the rule for changing a mixed number to an improper fraction. This approach focuses students’ attention on replicating a procedure for changing one form to another, specifically: *To change the mixed number $2\frac{3}{4}$ to an improper fraction, multiply the denominator 4 by the whole number 2 and add the numerator 3, to produce the improper fraction $\frac{11}{4}$.* By comparison, an inquiry-based approach would emphasize why the rule works, as for example, by connecting numerical and pictorial representations, as shown in Figure 1.

<table>
<thead>
<tr>
<th>Mixed Numbers</th>
<th>Improper Fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1\frac{1}{2}$</td>
<td>$\frac{?}{2}$</td>
</tr>
<tr>
<td>$2\frac{3}{4}$</td>
<td>$\frac{?}{4}$</td>
</tr>
<tr>
<td>$3\frac{3}{8}$</td>
<td>$\frac{?}{8}$</td>
</tr>
<tr>
<td>$4\frac{1}{2}$</td>
<td>$\frac{?}{2}$</td>
</tr>
<tr>
<td>$3\frac{5}{8}$</td>
<td>$\frac{?}{8}$</td>
</tr>
</tbody>
</table>

Figure 1. Structured-Inquiry Activity for Generalizing a Rule

Comparing the numerical columns of the worksheet to the diagrams, students learn that for the case of $2\frac{3}{4}$, multiplying the denominator 4 by 2 represents two wholes cut into fourths for a total of $8\frac{3}{4}$. Then adding the remaining $\frac{3}{4}$ yields the improper fraction $\frac{11}{4}$.

Structured-inquiry is an opportunity to develop students’ reasoning and data-analyses abilities through questions that foster different types of thinking, e.g. pattern-finding, generalization, and attention to what changes and remains constant from one example to the next. A directive such as, “Explain in words the pattern you have discovered,” encourages students to connect the numerical, diagrammatic, and verbal representations of the targeted rule. For this particular structured-inquiry activity, the objective for students is to think *with* the fraction...
concept and to notice the equivalence of the mixed number and the improper fraction, as well as the rule for converting one form to the other.

**Structured-Inquiry Activities for Generalizing Mathematical Formulas and Relationships**

Figure 2 illustrates the design of a structured-inquiry activity for generalizing the area formula for rectangles from a data table.

**Area of Rectangle Discovery:** Using the smallest square on the geoboard as the unit of area measure, identify and record the length, width, and area of each rectangle in the table. In the table below, length refers to the horizontal measure and width refers to the vertical measure.

<table>
<thead>
<tr>
<th>Rectangle</th>
<th>Length (in units)</th>
<th>Width (in units)</th>
<th>Area (in sq. units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>D</td>
<td>6</td>
<td>2</td>
<td>12</td>
</tr>
</tbody>
</table>

Explain, in words, relationships you see among the three columns of the table.

Figure 2. Structured-Inquiry Activity for Generalizing a Mathematical Formula

It is assumed that students do not yet know the formula for area of a rectangle, but are able to figure out the area of rectangles drawn on geoboard-dot paper. For rectangular regions whose sides are vertical and horizontal with respect to the \( x \)- and \( y \)-axes, determining area by subdividing a given rectangle is a straightforward partition into square units.

Once students record the measurements for length, width, and area of each rectangle in the data table for Figure 2, they look for patterns in the data by responding to the given direction: *Explain in words relationships you see among the three columns of the table.* Close analyses of the data by students in small groups and through whole-class discussion leads students to the generalization of the formula for area of a rectangle; *area = length \times width.* With this development, students are now able measure the areas of rectangles in two ways: 1) by counting unit squares on the geoboard dot paper, and 2) by using the area formula for rectangles. Subsequent practice activities ideally would ask students to find the area of a specific rectangle in these two ways to maintain the link in students’ minds between the spatial interpretation of area measurement and its algebraic/numerical representation.
Structured-Inquiry Activity for Generalizing a Concept

Next is a structured-inquiry example for learning new concepts in which students experience three instances of the concept. After each experience, the teacher introduces the name of the concept. Based on these three examples (Figure 3), students generalize the meaning of the concept and write a definition. Through interactive whole-class discussion, the teacher guides students to tweak their definitions to be mathematically correct.

Figure 3. Structured-Inquiry Concept Development Activity

This exemplar structured-inquiry activity centers on the area concept through a floor-tiling setting. While distributing 4-by-6 index cards to students, one per student, the teacher sets up the problem context: “This card represents the kitchen floor in your home. You want to upgrade your kitchen by re-tiling the floor. How many color tiles does it take to cover the floor?” Students cover the index card with color tiles (Figure 3). When tiling is complete, the teacher introduces the name of the concept just experienced by writing, and stating aloud, “It takes ___ color tiles to cover the card, therefore we say:

The area of the card is ___ color tiles.”

Students’ second experience with the area concept requires covering the floor with a different-size tile: “How many post-its does it take to cover the floor?” Students cover the card with post-its and write: “It takes ___ post-its to cover the card, therefore we say:

The area of the card is ___ post-its.”

Next, a non-example provides additional insight into what area is and is not: “How many circles does it take to cover the 4 x 6 index-card floor?” Since the circles do not cover the card completely, students must refine their understanding of the concept: “It takes ___ (24) circles to ___ (almost) cover the card, therefore we say:

The area of the card is ___ (more than) ___ (24) circles.”
In small groups, students write a definition of area, share it with their small group and write a single group-definition. Through teacher-led whole-class discussion, student groups tweak their definitions to achieve mathematical accuracy. For example “area is the space within a flat plane,” or “area is the number of units it takes to cover the surface of a two-dimensional object, or “area is the space inside a flat closed surface.” Discussion leads to the distinction between area as a spatial concept (i.e., the space within a two-dimensional closed figure) and area measurement (the number of units covering that space). In this activity, the concept focus is on what is being measured—the two-dimensional space within a closed figure.

Closing

Understanding and acceptance of inquiry-based mathematics instruction is growing. I am always impressed when my pre-service students desire to implement the new perspective on mathematics teaching in their future classrooms. The Common Core State Mathematics Standards (NGA & CSSO, 2010) and assessments help to move practicing and aspiring mathematics teachers’ thinking forward in the direction of inquiry. YouCubed, the popular informative website of Stanford Professor Jo Boaler, provides examples of inquiry-oriented mathematics thinking that contrasts with the prevailing direct-instruction approach. Logical flow is the essential difference. Whereas inquiry-based mathematics teaching typically employs inductive teaching methods, direct instruction is characteristically deductive. The challenge, as previously mentioned, is the perception that inquiry takes time—time that teachers perceive they do not have due to, among other things, the pressures of high stakes testing and a topic-heavy curriculum. In this article, I have illustrated ways that structured-inquiry activities might address the complaint of too-little time while also addressing the standards mandate to teach mathematics for robust understanding (Schoenfeld, 2016; NGA & CCSSO, 2010).

References


Barbara Kinach, Ed.D., is Associate Professor of Mathematics Education at Arizona State University. She teaches mathematics methods courses for elementary, middle, and high school pre-service teachers at both the undergraduate and graduate levels. Kinach’s research focuses on the impact of methods-course tasks on pre-service teachers’ (often procedural) understanding of mathematics and preferences for direct-instruction teaching methods. Through design research, she is developing methods-course tasks targeting: 1) inductive vs. deductive learning-task designs for mathematical concepts, relationships, formulas, and rules, 2) figural sequences for facilitating students’ transition to symbolization in algebra, and 3) concrete-pictorial-abstract lesson sequencing. The central role of visualization and representations of mathematics content in pre-service teacher learning can be seen in her work, especially: “A Cognitive Strategy for Developing Pedagogical Content Knowledge in the Secondary Mathematics Methods Course: Toward a Model of Effective Practice” (Teaching and Teacher Education, 2002), and “Progressive Visualization Tasks and Semiotic Chaining for Mathematics Teacher Preparation: Towards a Conceptual Framework” (Signs of Signification: Semiotics in Mathematics Education Research, 2018). As principal investigator for a grant with The MIND Research Institute, her research will be expanding to include the impact of curricular tasks from Spatial Temporal Mathematics (ST Math) on children’s mathematics concept development.
Is the Order of Operations Prescribed by Some “Higher Power”?

Fabio Augusto Milner

Abstract

The need for agreement on order of operations brings to light some of the facts that are frequently ignored or hidden from students and teachers. Among them, the simple truth that the use of parentheses is enough to suppress any need for an agreement on order of operations; the fact that any agreement on order of operations is based on the goal of minimizing the use of parentheses; the role that the associative property of an operation plays in order of operations; as well as journal articles that give false information about the topic. Activities are provided to guide students to understand what parentheses are rendered unnecessary by an agreed upon order of operations, as well as when to place necessary parentheses. An activity is provided to bring to light the importance of parentheses when using division as multiplication by the reciprocal under the standard US order, multiplication-before-division.

When we say “order of operations,” the most frequent reactions are cringing or thinking PEMDAS (Parentheses, Exponentiation, Multiplication, Division, Addition, Subtraction). What does “order of operations” mean? It is really quite simple. We have several operations to perform and we have to indicate the order in which they should be performed. That is all there is to order of operations.

Let us focus on understanding the problem when considering two binary (i.e. performed on two numbers) operations, addition and multiplication. We need three numbers, as for example, \( a, b \) and \( c \), and we want to assign a meaning to \( a + b \times c \). Forget PEMDAS (forget cringing too because this is simple!). We have two different ways of calculating \( a + b \times c \):

1. \( (a + b) \times c \)
2. \( a + (b \times c) \)

If we choose \( a = 2, b = 3 \) and \( c = 5 \), then these result in:

1. \( (2 + 3) \times 5 = 25 \)
2. \( 2 + (3 \times 5) = 17 \)

These different answers (25 and 17) depend only on the order in which the operations are performed. Addition-before-multiplication produces 25. Multiplication-before-addition produces 17. If we have to perform several operations and we agree to indicate each operation by placing parentheses around the two numbers involved and separated by the corresponding operation symbol, then the use of one pair of parentheses for each operation removes all ambiguity. Actually, there is no need for parentheses for the last operation because there are no choices left for it.

In summary, in order to perform \( n-1 \) binary operations on \( n \) numbers, we can avoid any ambiguity by using \( n-2 \) parenthesis pairs, with exactly one of the operations inside each of the
parentheses. Each of the numbers inside parentheses is of one of the following types: original numbers from the initial \( n \), or new numbers obtained as the result of one of the operations performed on a combination of previously computed numbers or initially given numbers. For example, for the five numbers 2, 3, 5, 7 and 11, connected by three additions and one multiplication, the computation that we usually indicate as \((2 + 3) \times (5 + 7) + 11 = 71\) would actually be written using three parenthesis pairs as \((2 + 3) \times (5 + 7) + 11 = 71\). In this last expression, the left side \(((2 \, \star \, 3) \, \diamond \, (5 \, \star \, 7)) \, \star \, 11\) is unequivocally and uniquely defined for any operations \( \star \) and \( \diamond \), whether or not they are associative, as we shall see momentarily. Note that, without parentheses but using the US-agreed-upon order of operations, the expression would result in \(2 + 3 \times 5 + 7 + 11 = 35\), a different number.

It may seem like we still need to somehow indicate the order in which the single operations inside any (same-colored) pair of parentheses are to be performed. Fortunately, that is not the case. In the example, we cannot perform the \( \star \) operation between the number in the red parentheses and the number 11 before we perform the \( \diamond \) operation inside the red parentheses, because we need to know what number the red parentheses encloses in order to perform \( \star \) on it and the 11. Likewise, we may not perform the \( \diamond \) operation inside the red parentheses before performing the \( \star \) operations inside the blue and the black parentheses. As for the latter two \( \star \) operations, we may perform them in either order (blue-before-black or black-before-blue), because the number each of them produces is irrelevant to the computation of the other. To provide guidance on how to avoid having any choices to make, we then perform those from left to right, blue-before-black in this case. It is important to stress that this is not necessary to avoid ambiguity, but just to provide an algorithmic way of performing the \( n - 1 \) operations. As far as avoiding ambiguity, the \( n - 2 \) parenthesis pairs are all that are needed, one for each operation, except for the last operation that needs no parentheses.

There is confusion among teachers when discussing order of operations, and what is dictated by properties of operations, and what by choice. One good example of such confusion is the article (Bay-Williams and Martinie, 2015, p.20) claiming that it is a myth that the order of operations was arbitrarily designed long ago. The authors say, “On the contrary, the order in which calculations are made has a firm mathematical basis in any era. Students should understand why multiplication precedes addition in expressions like this one: \(4 + 3 \times 5\). If the 4 is added to the 3 before multiplication, the answer is quite different – and incorrect.” De Villiers (2015, p.9) points out the obvious, namely that if we (as the authors do) assume that multiplication is to be performed before addition, then we can see that the opposite order leads to a wrong result, a good example of circular reasoning. We urge caution about the false claims made therein, lest we perpetuate common misunderstandings being passed on to more teachers and their students.

We should distinguish unary from binary operations when discussing their order. A unary operation takes one number as input and produces one number as output. The simplest of unary operations, \( \text{next} \) (for whole numbers, \( \text{next}(n) = n + 1 \)), is not conceived as an operation in
schools, even among teachers. Yet it is very intuitive from a very young age and fluently practiced by most early childhood learners. In the upper elementary grades, the only unary operations introduced as operations are squaring and cubing. For two whole numbers \( a \) and \( b \) in that order in \( a^2 \), \( b = 2 \) is squaring and \( b = 3 \) is cubing. A binary operation takes an ordered pair of numbers as input and produces a number as output.

In K-12 education, binary operations are introduced early in the elementary grades, beginning with addition of natural numbers. Young children learn how to add two natural numbers by “counting on,” and when they reach some level of fluency with addition of two natural numbers, they are presented with the problem of adding three or more numbers, tacitly giving rise to the issue of order of operations. In an article on teaching and learning addition and subtraction, Fuson (1992, p. 57) states: “Preschool children as young as 3 and 4 years show some understanding of very simple cardinal addition and subtraction situations in which addition is viewed as “getting some more” and subtraction is viewed as “losing some”.”

The associative property of a binary operation already makes many parentheses superfluous. For example, \( 2 + 3 + 5 + 7 + 13 \) contains five numbers and four operations, and under the general discussion presented above, would require three pairs of parentheses to avoid ambiguity in the absence of an order of operations agreement. However, the associative property makes all three pairs of parentheses superfluous because it removes all ambiguity even in the absence of any agreement about order of operations: \( (2 + 3) + (5 + (7 + 13)) = 30 \), just as \( ((2 + 3) + 5) + 7 + 13 = 30 \) and \( 2 + (3 + (5 + (7 + 13))) = 30 \), and so does any other grouping. Everything in this paragraph applies to multiplication because multiplication is also associative. By contrast, it does not apply to the power-operation because it is not associative (i.e. \( a^{(b^c)} \) is not equal to \( (a^b)^c \) unless \( b^c = bc \)). Therefore, the computation of more than one power operation needs either a rule or parentheses to decide the order in which they are to be performed, that is, either prescribing that \( a^{b^c} \) means \( a^{(b^c)} \) by agreement (rule), or always using the parentheses that do away with the ambiguity.

We can now give a definitive answer to the question posed in the title of this article: NO, if we understand “higher power” to mean mathematical reasoning. The order of operations is an arbitrary choice, and actually, an unnecessary one as far as the computations are concerned: Using \( n - 2 \) parenthesis pairs for \( n - 1 \) operations on \( n \) numbers removes the need for any supplementary information about the order in which those \( n \) operations are to be performed.

The raison d’être for prescribing some order of operations is an economic one rather than a mathematical one. The goal then becomes: Use as few parentheses as possible in order to avoid writing more than are minimally necessary to avoid ambiguity. Once we adopt an order of operations agreement, we can evaluate its efficiency in avoiding the need for some parentheses. For example, we can do this by taking an arbitrary expression with \( n \) numbers related by \( n - 1 \) operations of addition and multiplication, whose order is expressed using \( n - 2 \) pairs of parentheses, and counting the number of pairs of parentheses that may be removed without
changing the result of the complete computation under the particular order of operations agreement. Such parentheses are called superfluous or unnecessary. In the expression \((2 \times 3) + (4 \times 5)\), for example, all three pairs are superfluous under the usual order of operations.

The following activities should help solidify some of the points made in the preceding discussion.

**Activity 1:** Let us adopt addition-before-multiplication for order of operations and use the standard symbols for both. Find the values of:

a. \(5 \times 2 + 3\)
b. \(2 \times 3 + 4 \times 5\)
c. \((2 \times 3) + 4 \times 5\)

**Activity 2:** Using addition-before-multiplication, which parentheses are unnecessary?

a. \(((2 \times 3) + 4) \times 5\)
b. \((2 \times ((3 + 4) \times 5))\)
c. \(((2 \times 3) + (4 \times 5))\)

**Activity 3:** Define, as usual, division by \(b \neq 0\) as multiplication by the reciprocal of \(b\). Find the errors in the following calculations using multiplication-before-division and fix them by inserting necessary parentheses.

a. \(2 \times 3 \div 5 = 2 \div \frac{1}{3} \times \frac{1}{5} = 2 \div \frac{1}{15} = 2 \times 15 = 30\)

b. \(2 \div 3 \times 5 = 2 + 3 \times \frac{1}{5} = 2 + \frac{3}{5} = 2 \times \frac{5}{3} = \frac{10}{3}\)

c. \(2 \div 3 \times 5 = 2 \times \frac{1}{3} \times 5 = \frac{10}{3}\)

To summarize, it is likely that the adopted agreement came to be multiplication-before-addition because of the economy this choice provides in the most basic property connecting the two operations, the Distributive Law: it is multiplication that distributes over addition, not vice versa. Thus, we may write the property as \(a \times (b + c) = a \times b + a \times c\) using only one pair of parentheses, while under the agreement addition-before-multiplication, the distributive property would be \(a \times b + c = (a \times b) + (a \times c)\), requiring twice as many parentheses. However, this example also points to an even more important consideration when performing many operations: It is not how many parentheses we need to use, but rather how many operations we need to perform. It is much more efficient to compute:

\[13 \times 3 + 13 \times 5 + 13 \times 7 + 13 \times 11 = 39 + 65 + 91 + 143 = 338\] (no parentheses needed under agreed-upon order, but using four multiplications and three additions) after applying the Distributive Property: \(13 \times (3 + 5 + 7 + 11) = 13 \times 26 = 338\) (one parenthesis pair and still three additions, but just one multiplication!). This is the message we all need to understand and learn.
References


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Constructing Triangles with Technology: Modeling Important Concepts in Geometry

Benjamin Sinwell and Nicole Bannister

Abstract

This article presents three related activities and how they were used in several high school geometry classes. Using Google Maps and GeoGebra, students constructed and measured triangles that were created from locations chosen by them (one per student). The triangles were then used to launch the three activities about: 1) angle measure and similarity; 2) trigonometry and area; and 3) the Law of Sines and the Law of Cosines. Because triangles were created by the students, the students were motivated to make and explore connections between the different topics, and to use technology to model the mathematics.

Introduction

Using dynamic geometry software as a tool “helps students make sense of mathematics, engage in mathematical reasoning, and communicate mathematically” (NCTM, 2014, p.78). The activities described in this article show how students in a high school geometry class generate their own triangles and then investigate angle measure, similarity, area, trigonometry, the Law of Sines, and the Law of Cosines.

Angle Measure and Similarity Activity: Directions

Using Google Maps, you will construct a triangle. Then using GeoGebra, you will measure the sides and angles of your triangle.

1. Use Google Maps to locate our school.
2. Choose two other locations, as for example, your: home, grocery store or some other location you visit.
3. Use the “measure distance” tool in Google Maps (right click to find the tool) to connect and measure the distance between the three locations (the school and the two other locations of your choice).
4. Do connections between the three locations form a triangle? If not, please change one of the locations (not the school). This is the Google Maps triangle.
5. Take a screenshot of your triangle and import it into GeoGebra (www.geogebra.org) using the “Image” tool ( ) under the “Media” section.
6. Create a text box and use it to describe the three locations that form your triangle.
7. Construct a triangle in GeoGebra using the three locations as vertices. This is the GeoGebra triangle.
8. Measure the angles and sides using the appropriate tools ( ) under the “Measure” section.
9. Save your work.

Answer the following questions:

• How do the lengths of the three sides of the GeoGebra triangle compare to the three distances given by the Google Maps triangle?
• Are the Google Maps triangle and the GeoGebra similar triangles? How did you decide?

Figures 1 and 2. Student Work Sample
The mathematical content goal of this activity is for students to solve mathematical problems in a real-world context involving angle measure and similarity. The mathematical practice goals for this activity are for students to model with mathematics and use appropriate tools strategically. The pedagogical goals for this activity are:

- Have students create their own mathematical adventure (creating their own triangles).
- Engage students in the use of dynamic geometry software (GeoGebra).
- Assess student understanding of angle measure and similarity.

Students were very excited about being able to choose locations that were relevant to their lives, and to use them to create mathematical models to investigate. Although, students had different levels of expertise using GeoGebra and Google Maps, they were able to communicate with each other (and with the teacher) to figure out how to construct their triangles and to measure sides and angles. After students completed the questions at the end of the activity, they submitted their answers to the teacher and participated in a whole group discussion.

Students noticed that the sum of the measures of the angles of the triangles were 180 degrees, and wondered about whether or not those triangles were congruent. They also noticed that the corresponding sides of the two triangles were in proportion. When students talked about whether or not the two triangles were similar, they debated about whether or not the Google Maps triangle was a scale model representing the actual distances (in centimeters or inches), or if the triangle represented one that displayed the actual distances (in miles). Students generated two answers based on different assumptions: 1) the two triangles are congruent (and similar) if the Google Maps triangle and the GeoGebra triangle were the same size, and 2) the two triangles are similar (not congruent) if the Google Maps triangle is. The teacher followed up by asking students what it means for two triangles to be congruent and what it means for two triangles to be similar.
Trigonometry and Area Activity: Directions

The mathematical content goal of this activity is to use trigonometry to solve problems in real world contexts. The mathematical practice goal of this activity is to make sense of problems and persevere in solving them. The pedagogical goals of this activity are for students to: 1) “elicit and use evidence of student thinking” (NCTM, 2014, p.10) and to 2) “pose purposeful questions” (NCTM, 2014, p.10) that help them make connections between mathematical concepts and ideas.

You will be using trigonometry to find the height of your GeoGebra triangle. Next, you will find the area of the triangle.

1. Open the GeoGebra triangle that you constructed using three locations from Google Maps.
2. Figure out the height of your triangle.

Answer the following questions:

- How did you determine the height of your GeoGebra triangle? Describe your solution process.
- Use distances from Google Maps to estimate the height of the GeoGebra triangle. Explain your procedure.
- Estimate the actual height of the GeoGebra triangle. Why do you think that your estimate of the height is reasonable? Explain.
- What is the area of the GeoGebra triangle?
- What is the actual area, in square miles, represented by the Google Maps triangle?

This trigonometry and area activity was used after students had learned to use trigonometric ratios to identify side lengths and angles of right triangles.

Students began this activity by accessing a copy of the GeoGebra triangle they made in the previous activity. This immediately got students talking to each other about the locations they chose and how trigonometric ratios might apply to their own triangle. When trying to find the height using trigonometric ratios, some students asked if it mattered which side they chose as a base. Others asked if the height and base had to form a 90-degree angle. Once students understood that the height was a line segment through a vertex and perpendicular to a line containing the base, and were able to move forward, they realized that they could use the sine ratio to compute the height.

After determining the height, students described their reasoning process. Furthermore, students made connections to prior mathematical knowledge (e.g., similarity and proportional reasoning) to compute the height of the Google Maps triangle.
Law of Sines and Law of Cosines Activity: Directions

1. Choose three locations and locate them using Google Maps. List them below.
   - Location A: __________________
   - Location B: __________________
   - Location C: __________________

2. Use Google Maps to measure the distance using the “measure distance” tool (right click on the map) from Location A to Location B and from Location B to Location C.
   - Distance from Location A to Location B: ________ miles.
   - Distance from Location B to Location C: ________ miles.
   - Do NOT find the distance from Location A to Location C.

3. Measure angle ABC.
   - Angle ABC: ________ degrees.

4. Use the Law of Sines and the Law of Cosines to calculate the distance between Location A and Location C. Describe your solution process.

5. Use Google Maps to measure the distance between Location A and Location C. Is the distance you calculated in #4 reasonable? Explain.

6. The triangle below displays the distances, in miles, between Yuma High School, the Yuma School District One office, and the City of Yuma Parks and Recreation Department. The smallest angle in the triangle below is 22 degrees. Find the measures of the other two angles. Show the steps of your solution.

The Law of Sines and the Law of Cosines activity was used after students had studied them. The mathematical content goal of this activity is to apply the Law of Sines and the Law of Cosines to non-right triangles in a real-world context. The mathematical practice goals of this activity are to model with mathematics and to use appropriate tools strategically. The pedagogical goals of this activity are to 1) “elicit and use evidence of student thinking” (NCTM, 2014, p.10), and 2) to increase student engagement and motivation by drawing on students’ prior knowledge and prior experiences.
Students were highly motivated to apply the Law of Sines and the Law of Cosines using Google Maps and GeoGebra. They commented that using their skills with technology and applying those Laws in a context that they had investigated previously, really helped them to understand the concepts behind the procedures. Several students asked if they could create another triangle for more practice.

**Conclusion**

By having students create their own triangles using locations that are important or familiar to them, an environment was fostered to help students “feel empowered by mathematics” (NCTM, 2018. p.25). Making repeated connections throughout a course to a context which has meaning to students helps to motivate them to gain a deeper understanding of the mathematics.

**References**

GeoGebra: [https://www.geogebra.org/](https://www.geogebra.org/)

Google Maps: [https://www.google.com/maps](https://www.google.com/maps)


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Commentary:
Computational Thinking—It’s Time
Colleen Megowan-Romanowicz

Abstract
The use of computers in our daily lives is pervasive (iPhones, Alexa, iPods/iPads, desktop/laptop computers). This commentary describes why we should learn to communicate with our devices, at least at a basic level. The article uses familiar analogies (journey of an ant across a tabletop, learning English) to introduce basic concepts of coding.

If you could go back to school and study anything you wanted to study (assuming that time and money were not an issue, of course), what would it be?
I have asked myself this question many times in my life, and, in fact have gone back to school multiple times to study new things: biology teacher education, mathematics, physics, martial arts, and mathematics education research. I’ve learned a few things through self-study as well, including quilt-making, VPython, and html, sometimes for fun and other times through necessity. For a long time, I toyed with the idea of returning to school for a degree in earth science. (I’ve taught myself little bits, but a deeper broader understanding would make hiking, camping, and travel so much of a richer experience). In the last few years, something new has floated to the top of my list and it’s not going away: computer programming.

The first time I saw a personal computer (my mom’s new Apple II in the late 70s) I was afraid to touch it for fear that I might “break” it. I got my first personal computer in 1983. In 1993 I got my first two classroom computers: Mac SEs (with two floppy drives, no hard drive).
Twenty-five years later, I use at least four different computers every day (not counting the ones in my car). One fits in the front pocket of my jeans (an iPhone), another sits on the dining room table (Alexa). Then there’s the tablet on my nightstand and the laptop on my desk. And yet, I still barely know how to talk to them. I can only tell them what to do if I have the right software installed to interpret my needs to my device, so that it knows how to do what I want it to. Often it takes me multiple tries to really accomplish what I am trying to do. I should be more fluent by now. Would I live in Italy for 25 years and never learn any Italian? Not likely.

Computers are here to stay. They are a permanent fixture in our world and our students’ world. The user interface will change but the tool behind it will still function according to the same underlying principles. It’s time for us to bite the bullet and learn how to communicate with it. I’m not saying we all have to become programmers, but we should make it our business to at least understand, at a basic level, how to read and make sense of simple code (which is really step-by-step instructions that tell the computer how to do something) and how to write or edit these instructions, as necessary. And, we have to help our students begin to do this, as well. We learned arithmetic, didn’t we? And Algebra, and Geometry? And English—which, by the way is much more difficult than programming) We can do this. It’s not hard.
Consider the journey of an ant across a tabletop. Let’s assume its travelling in a straight line toward a sugar cube. When the ant crests the edge of the table it makes its way toward the sugar cube, that is 15 cm distant, at a steady rate of 2 mm per second. I could set up a time-lapse camera to take a picture of the ant once each second and then put these images together into a flip book. When I flip through it, the result will look something like this, right? (The x’s represent the successive positions the ant occupies as each picture is taken.)

How can we describe the ant’s changing position mathematically?

- initial - x = 0
- delta - t = 1
- next - x = x + 2

Now all we have to do is keep repeating this calculation until the ant reaches the sugar cube. That’s all there is to it. Simple addition, over and over and over again. It doesn’t really require higher level mathematics to do this, but if we did it by hand it would take us a l-o-n-g time. And by the time our ant gets to the sugar cube, we might be a little bored. It might take us longer to write down all the positions of the ant than it would take the ant to actually crawl across our tabletop. We might be tempted to change the speed of the ant to a not very realistic 10 mm per second, so that we don’t run out of paper (and patience) before it gets there.

Now, who wants to do this in their classroom? Come on, it’s easy. All you need is a jar of ants, some sugar cubes, paper, pencils and rulers. No? Did I hear you say No? Or perhaps “NO WAY!”

Computers excel (no pun intended) at repetitive calculations. What would you have to tell a computer to get it to do this for you?

- The initial position (initial - x).
- The amount by which this position changes each time the computer makes a calculation.
- How often to make a calculation.

The computer doesn’t know units so you would have to make sure your inputs (initial - x and delta - t) have the same units as your outputs. In the case of our ant, we would have to convert cm to mm but other than that we’re ready to go.
This is a computer program. You can read it, right, and predict what an output (and the next output, and the next output, and the next output…) might look like? It could output a table of values (positions and times) or a column of ants marching across the table. Why would you ask a student to do this? I can think of a number of reasons. How about understanding constant rate of change, \( \frac{dx}{dt} = r \cdot t \), or \( x = v \cdot \Delta t \) as we like to write it in physics where \( x \) is position, \( v \) is velocity, and \( \Delta t \) is a time interval? How about linking multiplication to repeated addition? How about mathematically modeling motion? How about taking first steps in learning how to really direct a computer to perform a task?

This particular code is written in a computer language called Pyret which was created to teach children mathematics, but its syntax and vocabulary are not that different from other computer languages. The red remarks preceded by hash marks (#) are just comments—annotations to help you (or for someone else who might read the code at some point and not know what the units of measure are) or tell you whatever else I might want you to know about what the code does. The computer only reads what is written on each line before a #. It ignores anything written after a #.

I have watched students get a little thrill every time they write a few lines of code like this that work. They don’t think it’s hard. When they start learning to write code, inevitably they remark that, “It’s about time we learned to do this. After all, we’ve already been using computers for years.”

Students think it’s no big deal, but do teachers? Not so much. It seems that some teachers are waiting for the other shoe to drop. For the last three summers I have been watching teachers learn to program like this in summer workshops. They react a lot like their students when their first program runs successfully: their arms go up in the air and they yell, “YES!” We spend 2+ weeks helping them learn to code in a variety of contexts involving motion and forces, and they become proficient.
After they leave the workshop and return to their classrooms, some can’t wait to show their students. Others can’t quite bring themselves to do so. It’s as though they are still waiting for that other shoe to drop.

Coding is not difficult. Sure, the syntax can be fussy, but then so is English. Do you remember getting red marks on your paper for writing a letter backwards, or misspelling a word, or forgetting to capitalize or punctuate? Writing code is no different. If students forget to type a colon where one is required, they get an “error message” (actually, they are not called error messages in the Pyret programming language, they are called “feedback”—a slightly gentler name). And if they forget to type the word “end” at the end of the program, they get feedback. Students are used to getting corrected on little details like this. They forgive themselves and type it over again. And after a little finger-practice they don’t make these errors anymore. Some teachers have a harder time forgiving themselves when they get feedback—why is that? We are too hard on ourselves sometimes, I think.

After a summer workshop, the difference between teachers who teach their students to program and those who don’t is not competence. They all leave with about the same competence. It’s confidence. It’s a lot like playing a sport that you know how to play. You play well or poorly because of what’s going on in your head, not because you don’t have the skills to hit the shots.

I’ve got news for you. There is no other shoe to drop. This is one of the easiest and most powerful learning opportunities you can opt for, and it will be a game changer for your students. Their mathematical thinking will be enhanced in ways we are only beginning to understand (for example, they will have a much easier time learning calculus when the time comes). The vocabulary is small, the syntax rules are few. It’s much easier than Italian (or Trigonometry).

As I write this, the annual “Hour of Code” event is happening. Next year, your students could participate if you just tell yourself YOU CAN DO THIS. If you think you can, you can, and…if you think you can’t, you can’t. It’s that simple.

Colleen Megowan-Romanowicz, Ph.D., taught middle and high school science and mathematics for almost 30 years. She returned to school at age 50 to study student thinking and learning, earning her Ph.D. in Physics Education Research from ASU in 2007. She taught and did research on STEM teaching and learning at ASU until her retirement in 2014. From 2011 through 2017, she served as executive officer of the American Modeling Teachers Association, a grassroots community of science teachers who use the Modeling Method of Instruction. She now serves AMTA as Senior Fellow. She continues to lead Modeling Workshops for teachers, and conduct research on student thinking and teacher leadership.
Evaluate Apps for Instruction, Practice, and Assessment
Carole Greenes, Mary Cavanagh, James Kim, Renee Ashlock, Lynda Guetter, Mari Westerhausen, and Tanner Wolfram

Abstract
As part of an NSF-funded project, high school teachers and students collaborated to evaluate existing free or inexpensive apps useful for instruction, data-collection and analyses, and assessment, in preparation for the creation of new apps. During the app evaluation process, teachers observed students exhibiting great interest in the content of the apps, perseverance in discovering ways that the apps could be used to help them, and enhanced communications with others. Teachers also learned a great deal about new apps from their students. This article describes the analyses of apps that focused on mathematics.

Background
With funding from the National Science Foundation (2015-2019, #1509105), we conducted the App Maker Pro (AMP) project with the goal of using software development in a Design Village strategy to: increase high school students’ interest in and success with STEM subjects, and update high school STEM teachers’ technology, mathematics-science-engineering design content knowledge, and their pedagogical expertise. Over a period of 1 ½ years teachers and students explored free or inexpensive apps (IOS or Android) and developed new apps that address problems in STEM and related fields. This article describes the app evaluation process, results of app analyses by AMP teachers and their own students at the end of the project, and makes recommendations for app use in Instruction, Data Collection/Analyses, and Assessment.

During the first session, while students were evaluating Village topic-relevant apps, we observed that they were very engaged, and highly motivated to think deeply about the content, the types of information provided, the ways in which that information was presented (e.g., graphically, in text, pictorially), and the usefulness of the information in terms of its applicability to the representation/solution of problems. Several teacher participants also noted the power of the app evaluation process to generate student learning, collaboration, discussion, and persistence.

Three high school AMP teachers (Biology and Robotics; Business Operations and Graphic Design; and Mathematics, Physics, and Space Science) collaborated with four AMP leaders to evaluate four math apps that had previously been identified as particularly useful in the teaching of STEM and related subjects. The high school teachers then conducted App Evaluation activities with high school students (grades 8 – 12) in several of their mathematics and science courses, as well as with teacher colleagues. This is their story.
### Apps and Their Evaluations

Initially, high school teachers suggested that their students learn about and evaluate the same four mathematics apps that they had done. In all classes, students announced that they already knew about those apps, and would prefer to evaluate apps that they selected! Some students did evaluate some of the four teacher-identified apps. Other apps evaluated dealt with a variety of STEM topics. Students eagerly approached the task, and as noted by teachers, they didn’t want to stop! Most students did additional app evaluations at home. Teachers were delighted to learn about some of the apps “discovered” by the students.

The evaluation rubric (Figure 1) requested that apps be identified as useful for Instruction, Practice, Data Collection/Analyses, or any combination of those, for each of the rubric categories: Relevance, Customization, Feedback, Usability, and Sharing. Each of the five categories had scores ranging from 4 to 0, corresponding to the letter grades, A to D, or Not Applicable (see top right of form). Not applicable generally refers to reference-types of apps that provide definitions of terms or sets of solved problems. Total score per app could range from 0 to 20. The numbers reported after the descriptions below the form are average scores, based on number of evaluators.

#### App Evaluation Rubric

Your Name (Print): ___________________________ Date: ___________________________

Teacher’s Name (Print): ___________________________ Topic: ___________________________

Rate each app you tested by giving it a score of 1 – 4 for Relevance, Customization, Feedback, Usability, and Sharing.

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<tr>
<th>Use: I Instruction</th>
<th>App name:</th>
<th>App name:</th>
<th>App name:</th>
<th>App name:</th>
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<tbody>
<tr>
<td>P: Practice</td>
<td>Use:</td>
<td>Use:</td>
<td>Use:</td>
<td>Use:</td>
<td>4 A grade</td>
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<tr>
<td>A: Data Collection/Analysis</td>
<td>Use:</td>
<td>Use:</td>
<td>Use:</td>
<td>Use:</td>
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<table>
<thead>
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<th>4</th>
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<tbody>
<tr>
<td>The app’s focus has a strong connection to the purpose for the app and appropriate for the user.</td>
<td>The app’s focus is related to the purpose for the app and mostly appropriate for the user.</td>
<td>Limited connection to the purpose for the app and may not be appropriate for the user.</td>
<td>Does not connect to the purpose for the app and not appropriate for the user.</td>
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<table>
<thead>
<tr>
<th>Customization</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>App offers complete flexibility to alter content and settings to meet user needs.</td>
<td>App offers some flexibility to alter content and settings to meet user needs.</td>
<td>App offers limited flexibility to adjust content and settings to meet user needs.</td>
<td>App offers no flexibility to meet user needs.</td>
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<table>
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<th>Feedback</th>
<th>4</th>
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<th>1</th>
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</thead>
<tbody>
<tr>
<td>User is provided specific feedback.</td>
<td>User is provided feedback.</td>
<td>User is provided limited feedback.</td>
<td>User is not provided feedback.</td>
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</tr>
</tbody>
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<table>
<thead>
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</thead>
<tbody>
<tr>
<td>User can launch and operate the app independently. The navigation is logical and simple.</td>
<td>User needs to have an expert show or model how to operate the app.</td>
<td>User needs to be cued each time the app is used.</td>
<td>App is difficult to operate or crashes often.</td>
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</table>

<table>
<thead>
<tr>
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<th>3</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Specific performance summary is saved in app and can be exported.</td>
<td>Performance data are available in app but exporting is limited and may require a screenshot.</td>
<td>Limited performance data are accessible.</td>
<td>No performance summary is saved.</td>
<td></td>
</tr>
</tbody>
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Figure 1. Project AMP App Evaluation Rubric
Apps Evaluated by Students, Teachers, and AMP Project Leaders

Below is a description of each app reviewed along with the evaluations. The N identifies the number of reviewers. Note: Two apps, Math Ref and SnapCalc were reviewed by both teachers/leaders and students.

**GeoGebra** is a graphing calculator app that plots data, graphs function, finds special points of functions (e.g., roots, minimum, maximum), produces geometric constructions, does regression and best-fit lines, as well as 3D graphing. Results can be easily saved and shared.

<table>
<thead>
<tr>
<th>Teachers/Leaders</th>
<th>GeoGebra N=9</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
<td>Customization</td>
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</tr>
<tr>
<td>Feedback</td>
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</tr>
<tr>
<td>Usability</td>
<td>3.25</td>
<td></td>
</tr>
<tr>
<td>Sharing</td>
<td>2.00</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>15.13</td>
<td></td>
</tr>
</tbody>
</table>

**Math Ref** is an online compendium with more than 1,400 formulas, figures, and examples to enhance understanding of concepts and computations in mathematics, physics, chemistry, and other related content areas. To assist with computations, various tools, including a unit converter, a quadratic equation solver, and a tool to compute triangle measurements, are provided. (**Math Ref Lite** is the free version of the app that does not have many of the computation tools.)

<table>
<thead>
<tr>
<th>Students</th>
<th>Math Ref N=16</th>
<th>Average</th>
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</thead>
<tbody>
<tr>
<td>Relevance</td>
<td>3.25</td>
<td></td>
</tr>
<tr>
<td>Customization</td>
<td>3.13</td>
<td></td>
</tr>
<tr>
<td>Feedback</td>
<td>2.94</td>
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</tr>
<tr>
<td>Usability</td>
<td>2.81</td>
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</tr>
<tr>
<td>Sharing</td>
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<td>Total</td>
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<table>
<thead>
<tr>
<th>Teachers/Leaders</th>
<th>Math Ref N=9</th>
<th>Average</th>
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<td>Customization</td>
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<td>Feedback</td>
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<tr>
<td>Usability</td>
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<tr>
<td>Sharing</td>
<td>0.89</td>
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<tr>
<td>Total</td>
<td>12.22</td>
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</tbody>
</table>
Mathspace is designed to replace mathematics textbooks at the middle and high school levels. The content is correlated with the Common Core State Standards for Mathematics (NGA & CCSSO, 2010), Grades 6 – 12, and includes more than 30,000 questions.

Teachers/Leaders

<table>
<thead>
<tr>
<th>Mathspace N=9</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relevance</td>
<td>3.38</td>
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<tr>
<td>Customization</td>
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<td>Feedback</td>
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</tr>
<tr>
<td>Usability</td>
<td>3.50</td>
</tr>
<tr>
<td>Sharing</td>
<td>1.75</td>
</tr>
<tr>
<td>Total</td>
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</tr>
</tbody>
</table>

Photomath solves mathematics problems from photos of the problems taken by the user, and provides step-by-step explanations of the solution process. The app also features animated instructions on how to solve those mathematics problems. For some problems, multiple methods are presented. The app reads both text and handwritten problems.

Students

<table>
<thead>
<tr>
<th>Photomath N=7</th>
<th>Average</th>
</tr>
</thead>
<tbody>
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<td>Relevance</td>
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<tr>
<td>Customization</td>
<td>3.00</td>
</tr>
<tr>
<td>Feedback</td>
<td>3.43</td>
</tr>
<tr>
<td>Usability</td>
<td>3.71</td>
</tr>
<tr>
<td>Sharing</td>
<td>2.43</td>
</tr>
<tr>
<td>Total</td>
<td>16.43</td>
</tr>
</tbody>
</table>
SnapCalc is a phone camera calculator that scans a computation (print or hand-written) and provides a step-by-step solution with explanations. Problems in all areas of mathematics (e.g., algebra, calculus) can be solved.

<table>
<thead>
<tr>
<th>Students</th>
<th>Teachers/Leaders</th>
</tr>
</thead>
<tbody>
<tr>
<td>SnapCalc N=20</td>
<td>Average</td>
</tr>
<tr>
<td>Relevance</td>
<td>3.45</td>
</tr>
<tr>
<td>Customization</td>
<td>2.80</td>
</tr>
<tr>
<td>Feedback</td>
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</tr>
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<td>Usability</td>
<td>3.40</td>
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<tr>
<td>Sharing</td>
<td>2.85</td>
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<td>Total</td>
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</table>

Comparison of Student and Leader/Teacher Evaluations of Math Ref and SnapCalc

As can be in the tables, the major difference in scoring by students and teachers occurred with the Math Ref app in the categories of Feedback (higher rating by students), Usability (higher rating by teachers), and Sharing (higher rating by students). The differences in ratings may be attributed to users’ interpretations of each category.

Differences of ratings in the Feedback category may be due to the “narrowness” of the information provided, that is feedback is limited to information requested by the user, with little or no detailed explanations.

As described in the rubric, Sharing refers to the ability to distribute performance summaries (scores, difficulties, successes) to others, not to the sharing of the site information. If the rubric is used again, we recommend that the Sharing category be renamed, Sharing Performance Results. The revised rubric is reproduced at the end of this article.

Using Apps: Benefits for Students

Types of apps vary from those that may be used for self-assessment; additional study; storage of frequently needed information for easy retrieval; tools for computations, data collection and display; the preparation of presentations; and the enhancement of communication. Each of these is described below.

**Self-Assessment:** From early in their education careers (elementary school), students stop asking teachers for help. They don’t want to “look dumb” and lose the respect of the teacher, as well that of their peers. The need for help is a privacy issue. Using apps, students can identify what they know and what is still unclear. They can take photos of parts of texts or other instructional materials that are difficult, and use that information to seek app assistance.

**Additional Study:** Several apps engage students in explorations that reinforce what they have been studying, as well as enrich their understanding by leading them into new applications or varying algorithms for solving problems.
Storage of Information: Students can save and maintain various types of information that will be needed immediately (e.g., formula, graph, text description, notes) for in-school and homework assignments and long-term projects.

Computation, Data Collection, and Display: Various tools are available to assist students with performing complex computations and graphing (e.g., scientific calculators); and collecting time-related (e.g., stopwatch/timer) and other measurement data.

Preparation of Presentations: Students can use photo and video apps to record work on projects. Using the voice recorder, students can record explanations to accompany the photos and videos in preparation for oral presentations. Stored material on the phone can be connected directly to a projector.

Communication: Phone apps allow immediate communication with other students and the teacher. Students can discuss ideas among themselves at school and at home, and contact experts for needed information.

Using Apps: Benefits for Teachers

As well as those listed above for students, there are additional values for the use of apps by teachers during instruction, assessment, and scheduling.

Instruction: Some information may be teacher-designed instructional materials and notes that can be shared regularly. Other instruction or essential information can be accessed by students with their mobile phones at any time (school, home). Differentiated instruction is enhanced with the use of apps. For students having difficulty, there are apps that provide review and practice of needed concepts, skills and reasoning methods. For other students, apps can engage them in a variety of applications and explorations that extend understanding of those concepts, skills and reasoning methods.

Assessment: Students can take teacher-designed or copied text quizzes and tests on their mobile phones, and submit completed assessments to their teachers. Teachers can then evaluate the assessments and send messages back to the students. Assessments can be saved and maintained by both the teachers and students.

Time Schedules: Calendar apps help teachers establish and maintain dates for homework submissions, assessments, and class presentations. These schedules can also be accessed by students.

Comments from Students and Teachers

All students interviewed loved using the apps. As they said, “These apps are great! Saves me lots of time getting information.” “The app helped me understand the steps to solving the problem.” “Now I don’t have to go to anyone for help. I can find help on my phone, anytime, anywhere!” “I don’t have to carry books around anymore!”
Although most teachers were in favor of having students use apps for finding needed information, conducting data collection and analyses, and communicating with others, there were teachers who were not in favor of having their students use technology in class or for homework.

Said teachers who liked the apps, “These apps help students with topics and computations where they need more instruction. Apps are private tutors!” “Having students use apps for measuring, computing, and developing graphs, enhances their interest in solving application problems and working on long-term projects.”

One teacher who was less in favor of having students use apps, claimed, “The apps use a different algorithm than the one I teach. Very confusing!” Said another teacher, “My students who are using these apps are getting As on their homework and failing the tests.”

For solving systems of linear equations, some apps and textbooks show the Elimination Method. Other apps and textbooks show Substitution, Graphing, or Matrix methods. To the teacher who claimed that apps use different algorithms than the one taught, it would be interesting to have students compare the different methods (algorithms). With regard to the teacher who claimed that students using apps were “getting A’s on homework and failing the tests,” it would be interesting to have the students figure out why they failed those tests. What did they not understand?

**Final Note:** Two apps that were not evaluated, despite the fact that all students and teachers were in agreement that those apps are outstanding complements to the teaching and learning of mathematics, are: **Kahn Academy** and **CrashCourse**. Both are available on Android and iOS devices, as well as on YouTube.
App Evaluation Rubric (Revised)

Your Name (Print): _______________ Date: _______________
Teacher’s Name (Print): _______________ Topic: _______________

Rate each app you tested by giving it a score of 1 - 4 for Relevance, Customization, Feedback, Usability, and Sharing.

<table>
<thead>
<tr>
<th>Use: P: Instruction</th>
<th>App name: Use:</th>
<th>App name: Use:</th>
<th>App name: Use:</th>
<th>App name: Use:</th>
<th>Score approximately corresponds to:</th>
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</thead>
<tbody>
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<td></td>
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Relevance

Customization

Feedback

Usability

Sharing Performance Data

Total

Scoring Rubric for additional clarification:

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<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relevance</td>
<td>The app’s focus has a strong connection to the purpose for the app and appropriate for the user.</td>
<td>The app’s focus is related to the purpose for the app and mostly appropriate for the user.</td>
<td>Limited connection to the purpose for the app and may not be appropriate for the user.</td>
</tr>
<tr>
<td>Customization</td>
<td>App offers complete flexibility to alter content and settings to meet user needs.</td>
<td>App offers some flexibility to alter content and settings to meet user needs.</td>
<td>App offers limited flexibility to adjust content and settings to meet user needs.</td>
</tr>
<tr>
<td>Feedback</td>
<td>User is provided specific feedback.</td>
<td>User is provided feedback.</td>
<td>User is provided limited feedback.</td>
</tr>
<tr>
<td>Usability</td>
<td>User can launch and operate the app independently. The navigation is logical and simple.</td>
<td>User needs to have an expert show or model how to operate the app.</td>
<td>User needs to be cued each time the app is used.</td>
</tr>
<tr>
<td>Sharing Performance Data</td>
<td>Specific performance summary is saved in app and can be exported.</td>
<td>Performance data are available in app but exporting is limited and may require a screenshot.</td>
<td>Limited performance data are accessible.</td>
</tr>
</tbody>
</table>
Carole Greenes, Ed.D., is Professor of Mathematics Education in the Ira A. Fulton Schools of Engineering and Professor and Director of the Practice Research and Innovation in Mathematics Education (PRIME) Center in the College of Liberal Arts and Sciences at Arizona State University. She is Principal Investigator for the NSF-funded App Maker Pro, Principal Investigator for the Helmsley Charitable Trust-funded Georgia Tech Vertically Integrated Projects program; and Director of the MATHadazzles book development project described in this article. Carole is author of more than 310 mathematics books and programs, 75 articles, five mathematical musical mysteries, and two histories of mathematics in story and song. She serves on the advisory board for the ASU Preparatory Academies, the STEM AZ Education Collaborative, the national MoMath, and the Math Museum in New York City. She is author of the online open source MATHgazine (grades 8 - 12) and MATHgazine Junior (grades 4 – 8). Carole is a 2018 recipient of the Lifetime Achievement Award from the National Council of Teachers of Mathematics and the MET Board. In October 2016, she received AATM’s Copper Apple Award for Leadership in Mathematics in Arizona.

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James Kim, M.A., is the Project Coordinator Sr. of the Practice, Research, and Innovation in Mathematics Education (PRIME) Center where he is responsible for the day-to-day operations, human resources, and project management. Currently, James manages the NSF grant-funded project App Maker Pro (AMP) and the Georgia Tech/Helmsley Charitable Trust-funded Vertically Integrated Project (VIP) at ASU. He also serves as the editor for the bi-annual OnCore Arizona Association Teachers of Mathematics (AATM) book, the 8-volume MATHdazzles Puzzle Books series, and the monthly MATHgazine published by the PRIME Center.

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Lynda Guetter teaches Business Operations and Graphic Design at Barry Goldwater High School and teaches. She is also an Adjunct Faculty member at Glendale Community College, and serves as an advisor for SkillsUSA and Arizona Chapter of Future Business Leaders of America.
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Tanner Wolfram, is Senior Project Assistant in the PRIME Center and senior in Barrett, The Honors College, at Arizona State University. The majority of Tanner’s work involves facilitating the NSF-funded Project AMP and the MATHadazzles puzzles books project. He co-authored Problems without Figures: 1909 and 2016, an article published in OnCore. He also works in the Barrett Mentoring Program, assisting freshmen with their studies. He is a member of the Society of Physics Students, a club for physics majors at ASU. Tanner enjoys learning about the U.S. government, and was formerly a part of the We the People competition. His team won the Arizona State Championship and competed at the National Competition in Washington D.C. In his spare time, Tanner enjoys learning about financial matters, especially the buying and selling of stocks.
Animations: Windows to a Dynamic Mathematics

Patrick W. Thompson

Abstract
Students have difficulty thinking of mathematics dynamically. Animations can be helpful in this regard, but only when the animations are designed to support teachers in holding conceptual conversations about important mathematical ideas.

It is a longstanding problem that students conceive mathematical ideas statically. We see this problem vividly in their understandings of variables. For example, students commonly think “x” in $3x^2 - 5 = 10 - x^2$ stands for the answer, when it is more productive to think, “Of all the values $x$ can have, which one(s) makes this statement true?” Or, more precisely, to understand the equation as, “Given $y_1 = 3x^2 - 5$ and $y_2 = 10 - x^2$. What value(s) of $x$ make $y_1$ have the same value as $y_2$?” This second way of thinking is behind understanding equations graphically (Figure 1).

![Figure 1. Graphical Illustration of $3x^2 - 5 = 10 - x^2$](image)
As teachers, we see many things in Figure 1 that are not evident to students who are learning about depicting equations’ solutions graphically. In particular,

- The variable $x$ in $3x^2 - 5 = 10 - x^2$ can have values that make the statement false. That’s how we get two graphs. If we limit ourselves only to values of $x$ that make the statement true (what students often call “the answer”), our graph would be composed of two points.

- Both graphs are composed of points having coordinates $(x, y_1)$ or $(x, y_2)$. Points of intersection tell us about solutions to $3x^2 - 5 = 10 - x^2$. But the intersection points are not solutions to the equation. Values of $x$ are on the $x$-axis. Values of $y_1$ and $y_2$ are on the $y$-axis. Intersection points have coordinates $(x, y_1)$ and $(x, y_2)$ so that $y_1 = y_2$ for the same value(s) of $x$.

We can help students see the nuances in Figure 1 by helping them envision a graph as emerging from the covariation of two variables (Moore & Thompson, 2015; Thompson & Carlson, 2017). This means they see the value of $x$ as varying, the value of $y_1$ as varying with the value of $x$, the value of $y_2$ as varying with the value of $x$, and see that all vary simultaneously.

However, it is difficult for students to understand diagrams like Figure 1 as depicting dynamic relationships. Instead, they commonly interpret graphs as if they are bent wire, associating their shapes with formulas having particular characteristics (e.g., “$x^2$” means bent up). You could use the animations linked here to help students develop ways of seeing dynamic relationships in static diagrams.

I must quickly emphasize that what students understand from animations depends greatly on the conversation their teacher manages around the animations. Students cannot easily decide on what to focus when several things happen at once. The teacher must bring these things to their attention. For example, in the animation linked above, it is incumbent upon the teacher to point out, for example:

- “Notice that values of $x$ are shown with a black bar and not a point. Why do you suppose the animator designed it like this?”
- “The value of $x$ starts to the left of zero. Do you see the value of $x$ getting larger or smaller as it varies toward zero?”
- “Notice the value of $y_1$ is shown with a bar along the $y$-axis. Why do you suppose it appears where it does? Did you expect it to appear somewhere else?”
- After showing values of $x$ and $y_1$ varying together, say “The animation’s title says, “The value of $y_1$ varies with the value of $x$. What does that mean? In what way does the value of $y_1$ vary with the value of $x$?”
• Before showing the graph of $y_1$ versus $x$, say “We see the values $x$ and $y_1$ varying simultaneously.” How could we anticipate the graph of $y_1$ versus $x$ by watching the two varying together? (e.g., Look in the plane using your peripheral vision to focus on a location of the correspondence point having values of $x$ and $y_1$ as coordinates.)

• After showing the graph of $y_1$ versus $x$ being generated, ask “What do you think is the purpose of those faint lines that meet where the graph appears?”

• Before showing the two graphs generated simultaneously, ask “Can you imagine both graphs being generated at the same time? What will the display look like while they are being generated? What will the display look like after they’ve been generated?”

• “What will be true when the values of $y_1$ and $y_2$ are the same for a value of $x$?”

The animations linked to this article and the questions above illustrate three important points.

1. For animations to be effective, students must attempt to anticipate what they will see before they see it, and then explain to themselves and to others what they have seen (Hegarty, Kriz, & Cate, 2003; Schnotz & Rasch, 2005).

2. The animation must be designed to support reflective classroom conversations (Cobb, Boufi, McClain, & Whitenack, 1997)—conversations that take students’ meanings and understandings as objects of discussion, as opposed to steps for getting answers.

3. Teachers must conceptualize the classroom conversation they wish to have. This includes important points to raise if they do not arise naturally. Moreover, the conversation must be organized around the mathematical ideas teachers wish students to learn (Thompson, 2002).

Animations can be an important aid in your instruction when your goal is to foster productive imagery. When learning new ideas or methods, students always learn more powerfully when they have imagery that helps them organize their activity. However, for animations to provide such support, you must think carefully about the mathematical thinking you hope to support.

Lastly, using animations productively in your instruction requires significant time and effort. Devote the energy to incorporate an animation into your instruction only if the mathematical ideas it supports are ones you anticipate students will use repeatedly in their future learning.
References


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**Pat Thompson, Ph.D.,** is Professor of Mathematics Education in the School of Mathematical and Statistical Sciences at Arizona State University. He teaches in the Math Education Ph.D. program and the secondary education Bachelor of Science program. Pat has researched mathematics teaching and learning at all educational levels. His most recent projects are to reform calculus curricula at ASU, and to investigate secondary school teachers’ mathematical understandings in both the U.S. and Korea.